OPERATING DEFLECTION SHAPES FROM TIME VERSUS FREQUENCY DOMAIN MEASUREMENTS

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ABSTRACT

The vibration parameters of a structure are typically derived from acquired time domain signals, or from frequency domain functions that are computed from acquired time domain signals. For example, the modal parameters of a structure can be obtained by curve fitting a set of frequency response functions (FRFs), or by curve fitting a set of (time domain) impulse response functions. Similarly, the operating deflection shapes of a structure can be obtained either from a set of time domain responses, or from a set of frequency domain responses.

Two of the most commonly asked questions about vibration are:

1. ***What is the deformation (deflection shape)** of a machine or structure under a particular operating condition?*
2. ***How much** is the machine or structure actually moving at certain points?*

Time domain responses can be used to answer both of these questions, for linear as well as non-linear vibration. On the other hand, frequency domain responses can be used to answer these questions for specific frequencies.

NOMENCLATURE

t = time variable (seconds).

ω = frequency variable (radians/second).

n = number of measured DOFs.

m = number of modes.

[M] = (n by n) mass matrix (force/unit of acceleration).

{x''(t)} = acceleration response n-vector.

[C] = (n by n) damping matrix (force/unit of velocity).

{x'(t)} = velocity response n-vector.

[K] = (n by n) stiffness matrix (force/unit of displacement).

{x(t)} = displacement response n-vector.

{f(t)} = excitation force n-vector.

{X(jω)} = discrete Fourier transform of the displacement response n-vector.

{F(jω)} = discrete Fourier transform of the excitation force n-vector.

[H(jω)] = (n by n) Frequency Response Function (FRF) matrix.

$\{x_f(t)\}$ = forced response n-vector.

$\{u_k\}$ = complex mode shape (n-vector) for the k^{th} mode.

p_k = pole location for the k^{th} mode = $-\sigma_k + j\omega_k$

σ_k = damping of the k^{th} mode (radians/second).

ω_k = frequency of the k^{th} mode (radians/second).

A_k = a non-zero scaling constant for the k^{th} mode.

$[R_k]$ = the (n by n) residue matrix for the k^{th} mode

$= A_k\{u_k\}\{u_k\}^{tr}$

tr - denotes the transpose.

***** - denotes the complex conjugate.

INTRODUCTION

The vibration parameters of a structure are typically derived from acquired time domain signals, or from frequency domain functions that are computed from acquired time signals. Using a modern multichannel data acquisition system, the vibration response of a structure is measured for multiple points and directions (DOFs) with motion sensing transducers. Signals from the sensors are then amplified, digitized, and stored in the system's memory as blocks of data, one data block for each measured DOF of the structure. A key requirement of the multichannel system is that it be able to ***simultaneously sample or digitize*** the vibration signals as it converts them from analog signals (voltages) to digital data (numbers).

If the acquisition system is also an FFT-based system (an FFT Analyzer), then one additional requirement must be met in order to compute valid frequency domain functions. To prevent ***aliasing*** (false frequencies in their frequency spectra) the frequency content of the time domain signals must be bounded to satisfy the Nyquist criterion. That is, the maximum frequency in the analog signals ***cannot exceed one half of the sampling frequency*** used to digitize them.

There are two ways to meet this criterion; digitize the time domain signals at a very high sampling rate (twice the highest expected frequency in the signals), or band limit the time domain signals with filters. Since the time signals must be frequency limited *before* they are digitized, they must be filtered as analog signals using analog filters. In a multichannel system, all of the anti-aliasing filters must filter all channels in the same way, and this usually adds to the cost of the system.

Having acquired either a set of sampled time domain responses, or computed (via the FFT) a set of frequency domain responses, an operating deflection shape is defined as:

Operating Deflection Shape: The values of a set of simultaneously sampled time domain responses at a specific time, or the values of a set of frequency domain responses at a specific frequency.

Operating Deflection Shapes versus Mode Shapes

The vibration response of a structure depends on both *the amount* and *the location* of its excitation. Therefore its operating deflection shapes *always* depend on its excitation source(s). On the other hand, the mode shapes of a structure do not depend on either the amount or location of its excitation. That is, they are *natural* properties of the structure, and will not change unless its physical properties (mass, stiffness, and damping), or its boundary conditions are changed.

Mode shapes, by themselves, do not have fixed values (they have no units), and therefore *cannot tell you how much a structure is actually moving*. They can only show the relative motions between two DOFs of a structure. Using modal parameters, the answer to the question, "How much ...?" can only be answered once the *amount* and *location* of all of the excitation forces is specified. Then, modal parameters can be used to synthesize an operating deflection shape.

But, what if the amount and location of all of the excitations cannot be identified, or are too complex to identify? Then, direct measurement of the operating deflection shape is the only way to answer the question, "How much ...?"

Since many vibration problems involve the excitation of modes (or resonances), operating deflection shapes and mode shapes must be closely related to one another, and indeed they are. In the following section, this relationship is examined in more detail.

THEORETICAL BACKGROUND

Since an operating deflection shape is defined simply as the response of a structure at a specific time or frequency, no assumption is made regarding the linearity of the structure's response. However, many vibration problems in structures involve the excitation of modes, which are only defined for linear systems. Furthermore, since the FFT is a linear transformation, it is much more profitable to begin the analysis of a vibration problem by assuming that the structure is behaving in a linear (or near linear) manner. Then, both time and frequency domain techniques can be used.

The equation of motion for a vibrating structure is commonly derived by applying Newton's second law to all of the DOFs of interest on the structure. In an experimental situation, this results in a countable set of equations, one for each measured DOF:

$$[M]\{x''(t)\} + [C]\{x'(t)\} + [K]\{x(t)\} = \{f(t)\} \qquad (1)$$

Notice that the excitation forces and responses are functions of time (t), and that the coefficient matrices **[M], [C],** and **[K]** are constants. This is a dynamic model for describing the vibration of a linear, time invariant structure.

The Fourier Transform

The Fourier transform is defined for continuous signals. The analogous discrete Fourier transform (DFT) is defined for discrete (sampled) signals. The FFT algorithm performs the DFT on a finite number of samples of time domain data. More specifically, the FFT transforms (N) samples of real valued time domain data into (N/2) samples of complex valued frequency domain data. Likewise, the Inverse FFT converts (N/2) samples of complex valued frequency domain into (N) samples of real time domain data. (For practical reasons, the number of samples (N) is usually restricted to an integer power of 2.)

When initial conditions are ignored, the equivalent frequency domain form of the dynamic model for a structure can be represented in terms of discrete Fourier transforms:

$$\{X(j\omega)\} = [H(j\omega)]\{F(j\omega)\} \qquad (2)$$

This equation is valid for all discrete frequency values for which the discrete Fourier transforms are computed. Taking the inverse FFT (FFT^{-1}) of the above equation yields the **forced response equation** for the structure. (For a fixed value of time (t), the forced response vector $\{x_f(t)\}$ is the **operating deflection shape**.)

$$\{x_f(t)\} = FFT^{-1}\{[H(j\omega)]\{F(j\omega)\}\} \qquad (3)$$

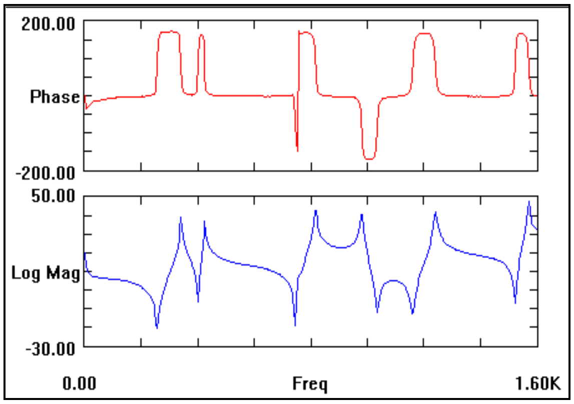

Figure 1. An FRF Measurement.

Several conclusions can be drawn from the forced response equation:

- The Fourier transform of the forced response is made up of a summation of the transforms of all of the excitation forces times the *columns* of the FRF corresponding to the excitation DOFs.
- The unit **Impulse Response** of the structure is obtained by setting one of the elements of the transformed excitation vector **{F(jω)}** equal to one (1), since the Fourier transform of a unit impulse is one for all frequencies. From the forced response equation, an n-vector of impulse responses is obtained by inverse Fourier transforming *one of the columns* of the FRF matrix, the column corresponding to the DOF where the impulse is applied. This column is also called the *reference* column.
- The impulse response of the structure depends on where the impulsive force is applied, which certainly serves our intuition.
- The **Sinusoidal Response** of the structure to a single sinusoidal excitation is obtained by inverse Fourier transforming the values of the *reference (excitation) column* of the FRF matrix, *at the sinusoidal frequency*. This is so because the Fourier transform of a sine wave signal is non-zero at the sine frequency, and zero for all other frequencies. The forced response equation also shows that for sine excitation, the operating deflection shape is merely the values of the FRFs from the reference column, at the excitation frequency.
- The forced response, (and hence the operating deflection shape), is completely arbitrary, depending on the combination of excitation forces acting on the structure.

Modal Parameters

If it is further assumed that reciprocity is valid for the test structure, (the [M], [C], and **[K]** matrices are symmetric), then the FRF matrix can be represented completely in terms of the modal parameters of the structure. Using superposition, the FRF matrix can be represented as a summation of terms, each term due to the contribution of a single mode of vibration:

$$[H(j\omega)] = [H_1(j\omega)] + [H_2(j\omega)] + \ldots + [H_k(j\omega)] + \ldots + [H_m(j\omega)]$$

where:

$$[H_k(j\omega)] = A_k\{u_k\}\{u_k\}^{tr}/(j\omega - p_k) + A^*_k\{u^*_k\}\{u^*_k\}^{tr}/(j\omega - p^*_k) \qquad (4)$$

Notice that each term of the FRF matrix is represented in terms of a pole location and a mode shape. Notice also that all the numerators are simply constants, and that only the denominators are functions of frequency. The numerators are also called **residues**. Each term of the FRF matrix can also be represented in terms of poles and residues:

$$[H_k(j\omega)] = [R_k] / (j\omega - p_k) + [R^*_k] / (j\omega - p^*_k) \qquad (5)$$

where:

$[R_k]$ = the (n by n) residue matrix for the k^{th} mode

$= A_k\{u_k\}\{u_k\}^{tr}$

Again, it's worth noting that the numerators (residues) of an FRF are merely constants, fixed in value. The mode shapes are *eigenvectors*; that is, they can change in value, but not in shape. The denominators are functions of frequency, and cause the peaks in an FRF. The locations of the peaks are dictated by the pole locations (p_k). Each peak in the FRF is evidence of at least one pole, or mode, or resonant condition. A typical FRF measurement (one element of the FRF matrix) is shown in Figure 1.

Forced Response in Terms of Modes

Several more conclusions can be drawn by substituting the modal parametric form of the FRF matrix into the forced response equation:

- Every element of the FRF matrix (between any pair of DOFs) is a summation of contributions from *all of the modes* of the structure. Therefore, the forced response potentially contains contributions from all of the modes.
- The operating deflection shape depends not only on the excitation forces, but also on the locations of the poles (resonant peak frequencies) and the structure's mode shapes.
- If an excitation force puts energy into a structure near a resonant peak frequency, the operating deflection shape could be very large, depending on the value of the

modal residue between the excitation and response DOFs.

- The modal residue between an excitation DOF and a response DOF is the *product* of the two mode shape components corresponding to the two DOFs. If *either* of the mode shape components is zero (on a nodal line of the shape), the mode will not contribute to the operating deflection shape.

The mathematics predicts what every vibration engineer knows from experience; namely, *if either the excitation or response DOF is on a nodal line of a mode shape, that mode will not contribute to the operating deflection shape.*

Impulse Response in Terms of Modes

Since the Fourier transform of the unit impulse response is one (1) for all frequencies, the impulse response n-vector due to an impulse applied at DOF(i) can also be written in terms of modal parameters:

$$\{Im_i(t)\} = FFT^{-1}\{[H(j\omega)]\{1_i\}\}$$
$$= \{Im_{i,1}(t)\} + \{Im_{i,2}(t)\} + .. + \{Im_{i,k}(t)\} + .. + \{Im_{i,m}(t)\}$$

where:

$\{1_i\}$ = n-vector with one (1) in the i^{th} element, zero elsewhere.

$$\{Im_{i,k}(t)\} = \{|R_{i,k}|\ e^{-\sigma_k t}\ \mathrm{sine}(\omega_k t + \alpha_{i,k})\}$$

= response of the k^{th} mode, due to an impulse applied at DOF(i).

$\{R_{i,k}\}$ = i^{th} column of the residue matrix for the k^{th} mode.

= $\{|R_{i,k}|\ e^{\alpha_{i,k}t}\}$ in polar form.

$e^{-\sigma_k t}$ = exponential decay envelope due to damping of the k^{th} mode.

$\mathrm{sine}(\omega_k t + \alpha_{i,k})$ = sinusoidal response at the frequency of the k^{th} mode.

This parametric form shows that the overall response of the structure to an impulse is the summation of the impulse responses of each of its modes. Each modal contribution is a damped sinusoidal response, with the oscillation frequency equal to the mode's natural frequency, and the decay envelope controlled by the mode's damping.

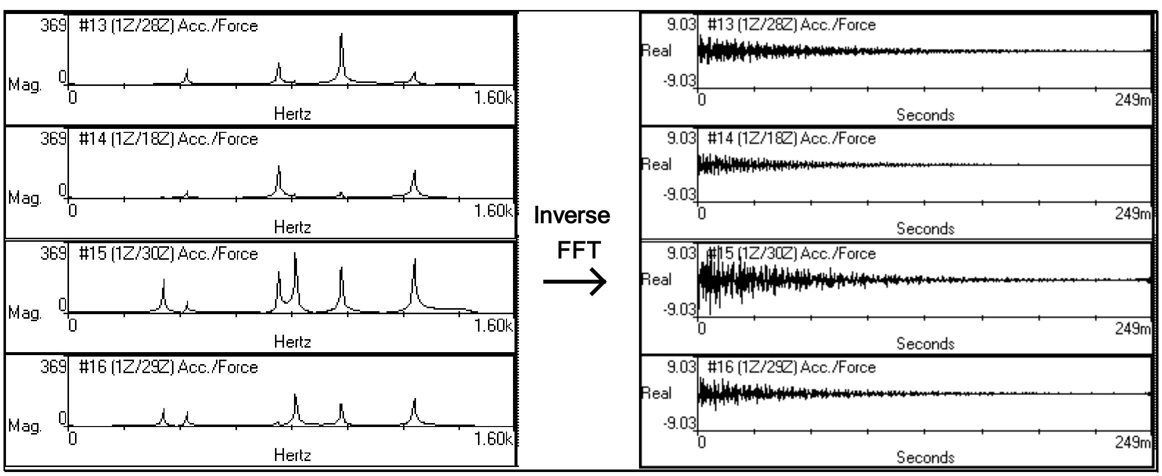

Figure 2. FRF Measurements and Corresponding Impulse Responses.

OPERATING DEFLECTION SHAPES FROM TIME DOMAIN FUNCTIONS

The operating deflection shape can be obtained directly from measured time domain responses of a structure, or from frequency domain measurements that have been inverse Fourier transformed.

Impulse Response

A set of impulse responses can be measured directly from a structure by simply impacting it and simultaneously sampling its responses at multiple DOFs. This is fast and direct testing method but requires a lot of simultaneously sampling acquisition channels if responses for a lot of DOFs are desired.

Alternatively, a set of impulse responses can be obtained by inverse Fourier transforming a set of FRF measurements. This second approach requires less equipment than the first approach because the FRF measurements don't have to be measured simultaneously. Only a 2 channel analyzer is required to make an FRF measurement. Since an FRF is a "normalized" measurement, computed, in effect, by dividing the transformed response by the transformed excitation, a set of FRFs can be measured one at a time. Hence, the requirement that all of the response signals be simultaneously sampled is relaxed. FRFs are typically measured one at a time during impact testing of a structure to obtain its modal properties.

Units of the Impulse Response

The units of an impulse response depend on whether it was acquired directly from the structure, or was computed by inverse Fourier transforming an FRF. If the impulse response was measured directly, and was calibrated, the response units are those of the vibration sensor; acceleration, velocity or displacement.

If the impulse response was obtained by inverse Fourier transforming an FRF, its units are acceleration, velocity, or displacement per unit of excitation force. For instance, if the FRF was measured using an accelerometer for response and a load cell for excitation, the impulse response units are *acceleration per unit of impact force at the impacting DOF*.

Figure 2 shows some FRF measurements, and their corresponding impulse responses, computed with the Inverse FFT. Figure 3 shows the mode shapes of the fundamental (lowest frequency) modes of a plate structure. Figure 4 shows 16 operating deflection shapes taken from the impulse responses, for 16.1 milliseconds following the impulse. These pictures show that all of the fundamental modes are excited from the impact point (reference DOF) on the corner of the plate, which is expected.

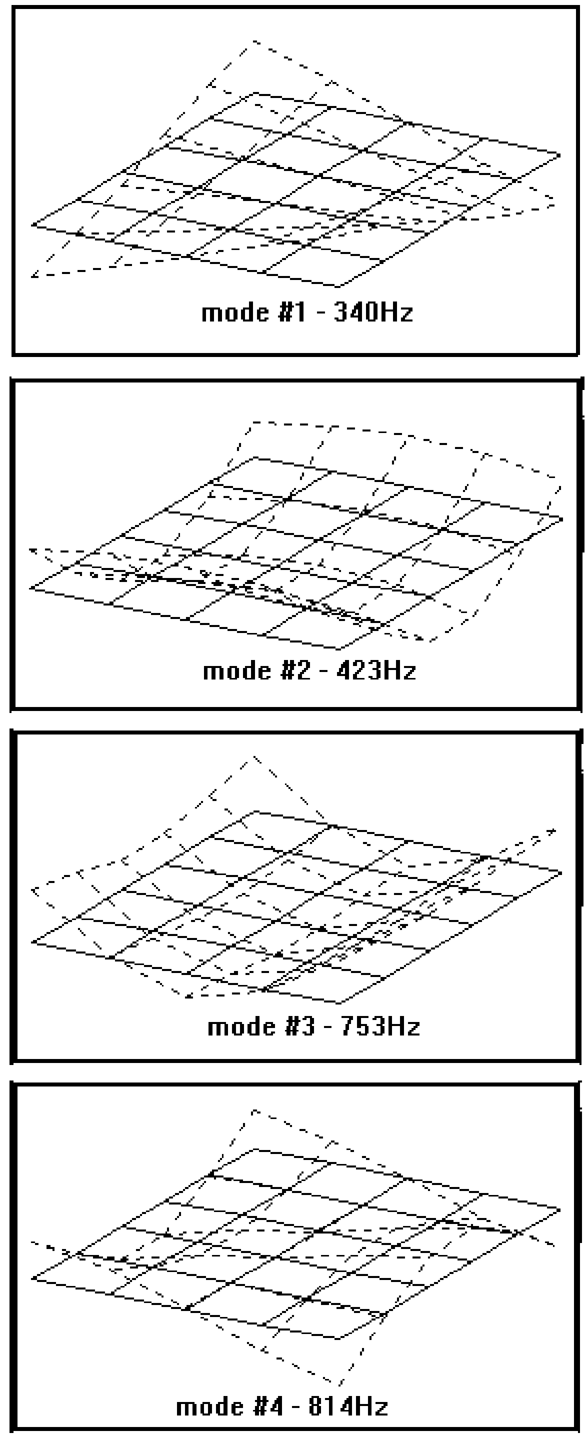

Figure 3. Mode Shapes of the Fundamental Modes.

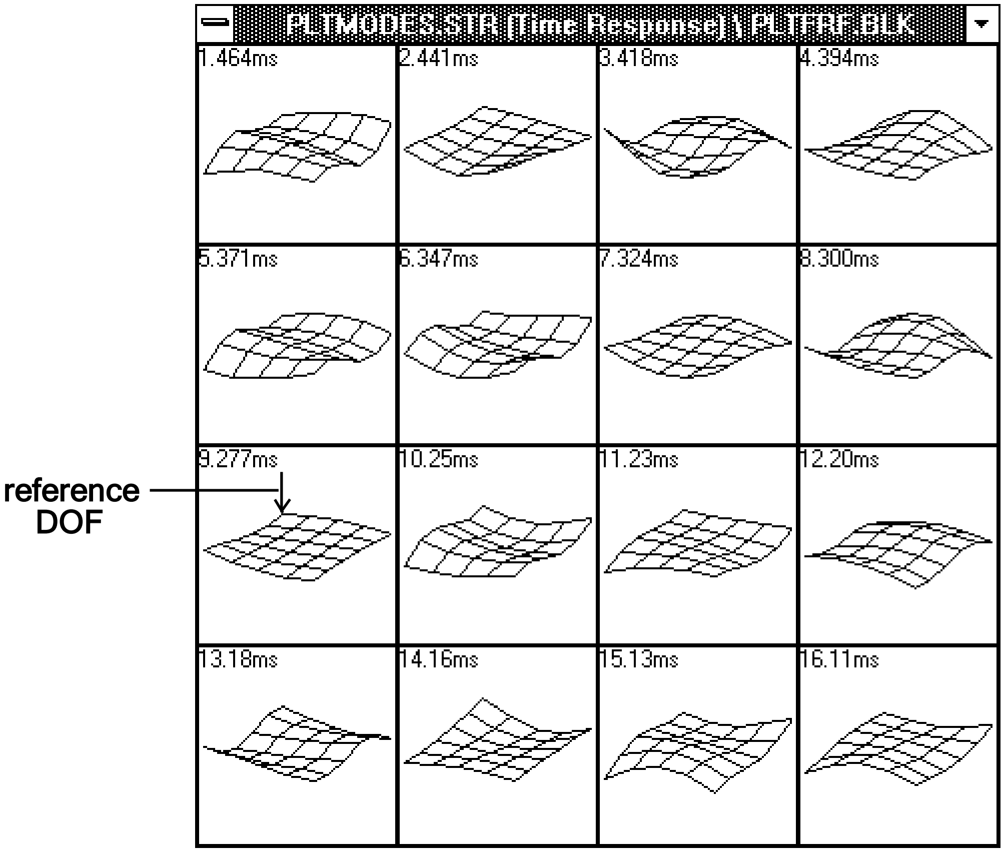

Figure 4. Operating Deflection Shape from Impulse Responses.

Sinusoidal Response

If the structure is excited by a single sinusoidal force, its steady state response will also be sinusoidal, regardless of the frequency of excitation. However, the *amplitude* of the response will depend on whether or not a mode (resonance) is also excited.

From the Theoretical Background section, we saw that in order to excite a mode, two conditions must be met:

- The excitation force must be applied at a DOF which is ***not on a nodal line*** of the mode shape.
- The excitation frequency must be "close" to the resonant peak frequency of the mode.

If both of these conditions are met, the mode will act as a "mechanical amplifier" and greatly increase the amplitude of response of the entire structure. This is commonly called a *resonance condition*. Conversely, if either condition is not met, the mode will not participate in the forced response of the structure.

OPERATING DEFLECTION SHAPES FROM FREQUENCY DOMAIN FUNCTIONS

Most modern FFT-based Analyzers can compute a variety of frequency domain functions that can be used for deriving operating deflection shapes, including the **Linear Spectrum (FFT)**, **Auto Power Spectrum**, **FRF**, and **Transmissibility**. Each of these measurement functions has certain advantages, depending on the test situation.

When any set of frequency domain functions is used to derive the operating deflection shape, the underlying assumption is that the deflection shape is the response of the structure if it were excited by a single sine wave at the reference DOF of the set of measurements. Frequency domain functions are therefore useful for examining how a structure would deflect if excited at the reference DOF, at any frequency within the bandwidth of the measurements.

In the Theoretical Background section, it was shown that sinusoidal excitation is equivalent to selecting the FRF values at the excitation frequency, from the column of FRFs corresponding to the reference DOF.

This is also true for the other types of frequency domain functions listed above. In other words, any set of vibration data taken from a structure is the result of applied excitation forces. Whether it be operating data, caused by self excitation, or data taken during a modal test, under tightly controlled excitation conditions, the operating deflection shapes are always subject to both the *amount* and *location* of the excitation.

Linear Spectrum

This frequency domain function is simply the FFT of a sampled time domain function. Phase is preserved in the Linear Spectrum, so in order to obtain operating deflection shapes from a set of Linear Spectra, the time domain signals must be simultaneously sampled. Since the Linear Spectrum is complex valued (contains both magnitude and phase information), the resulting operating deflection shapes will also contain magnitude and phase information.

Auto Power Spectrum

This frequency domain function is derived by taking the FFT of a sampled time domain function and multiplying the resulting Linear Spectrum by the complex conjugate of the Linear Spectrum at each frequency. Phase is not preserved in the Auto Power Spectrum, so a set of these measurements need not be obtained by simultaneously sampling all of the time domain responses. Since phase is not retained in these measurements, operating deflection shapes derived from them will contain only magnitude, and no phase information.

FRF

The FRF is a 2-channel measurement, involving a response and an excitation signal. It can be estimated in several ways, depending on whether the excitation or the response has more measurement noise associated with it. The most common calculation involves dividing the Cross Power Spectrum between the response and excitation signals by the Auto Power Spectrum of the excitation, at each frequency. Averaging of several Cross and Auto Power Spectra together is also commonly done, to reduce noise. Phase is preserved in the FRF, but a set of FRFs need not be obtained by simultaneously sampling all of the time domain responses. Each (response, excitation) pair must be simultaneously sampled, however.

Since a set of FRFs contains both magnitude and phase at each frequency, the operating deflection shapes derived from a set of FRFs will also contain both magnitude and phase information. The units of the operating deflection shapes are *acceleration, velocity, or displacement per unit of excitation force at the reference DOF*.

Transmissibility

Transmissibility measurements are made when the excitation force(s) cannot be measured. Transmissibility is a 2-channel measurement like the FRF. It is estimated in the same way as the FRF, but the response is "normalized" by a *reference response signal* instead of an excitation signal. Phase is also preserved in Transmissibility's, and a set of them need not be obtained by simultaneously sampling all of the time domain responses. Each (response, reference response) pair must be simultaneously sampled, however.

As with FRFs, a set of Transmissibility's contain both magnitude and phase at each frequency, so operating deflection shapes derived from a set of Transmissibility's will also contain magnitude and phase information. The units of the operating deflection shapes are *response units per unit of response at the reference DOF*.

Figure 5 shows the operating deflection shape taken from a set of FRFs for the plate structure at 780 Hz, in between the 3rd and 4th modes. Figure 6 shows the operating deflection shape for 760 Hz, close to the resonant frequency of the 3rd mode. Since, the motion at this frequency is dominated by the mode shape of the 753 Hz mode, the operating deflection shape "**looks like**" the 753 Hz mode shape, but its values will be different. Figure 7 shows the operating deflection shape for 800 Hz, close to the resonant frequency of the 4th mode.

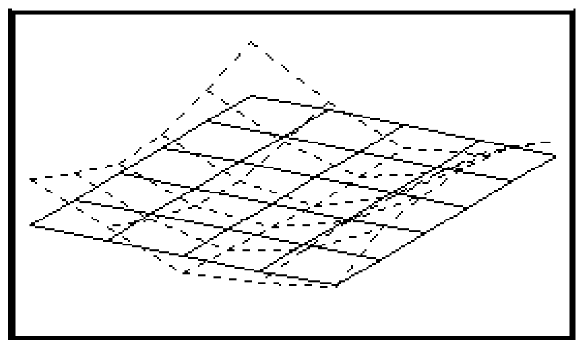

Figure 5. Operating Deflection Shape from FRFs at 780 Hz.

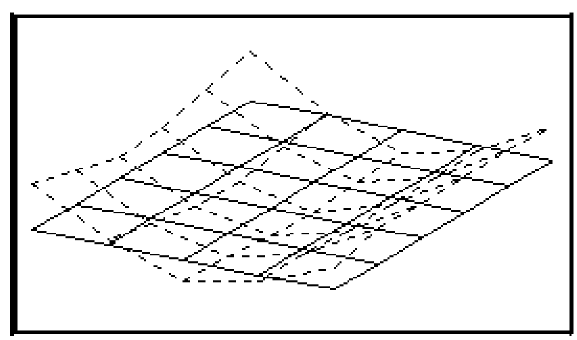

Figure 6. Operating Deflection Shape from FRFs at 765 Hz.

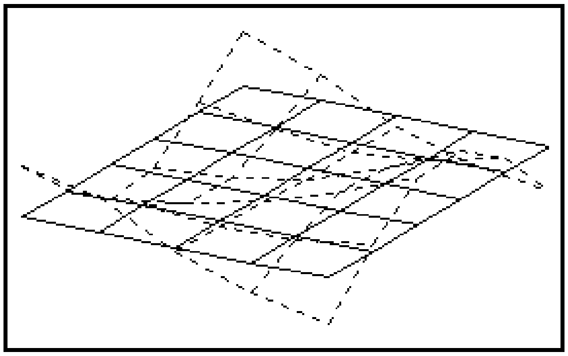

Figure 7. Operating Deflection Shape from FRFs at 785 Hz.

CONCLUSIONS

Operating deflection shapes were defined for both time and frequency domain functions. Operating deflection shapes and mode shapes were compared, and it was shown that they are quite different, yet related to one another.

- Operating deflection shapes depend on both the *amount* and the *location* of excitation forces on a structure, whether the forces are known or not. Modes shapes do not depend on excitation forces, but are *natural* properties of a structure.
- Operating deflection shapes show *how much* a structure is really moving. Mode shapes have no unique value, so they cannot be used directly to determine how much a structure is moving.
- Operating deflection shapes are functions of the modal properties (frequencies, damping, and mode shapes) of a structure. If the amounts and locations of excitation forces are known, then modal properties can be used to synthesize operating deflection shapes. For excitations close to the modal frequencies of a structure, its mode shapes will closely approximate its operating deflection shapes, but only in "shape", not in value.

There are many other details associated with manipulation of time and frequency domain measurement data, in order to animate and compare the operating deflection shapes and mode shapes of a structure. These will be presented in forthcoming papers.

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