MASS, STIFFNESS, AND DAMPING MATRICES FROM MEASURED MODAL PARAMETERS

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ABSTRACT

The theory of complex mode shapes for damped oscillatory mechanical systems is explained, using the matrix of transfer functions in the Laplace domain. These mode shapes are defined to be the solutions to the homogeneous system equation. It is shown that a complete transfer matrix can be constructed once one row or column of it has been measured, and hence that mass, stiffness, and damping matrices corresponding to a lumped equivalent model of the tested structure can also be obtained from the measured data.

INTRODUCTION

In recent years, there has been considerable activity in the study of elastic structure dynamics,. in an attempt to design structures that will function properly in a hostile vibration environment. Although much of the early work centered around fatigue and life testing, the latest efforts have been directed towards analytical modeling and simulation of mechanical structures. Distributed structures are generally modeled as networks of lumped mechanical elements, in an effort to predict failures more reliably and faster than is afforded by conventional life testing procedures.

With the advent of the inexpensive mini-computer, and computing techniques such as the Fast Fourier Transform algorithm, it is now relatively easy to obtain fast, accurate, and complete measurements of the behavior of mechanical structures in various vibration environments.

Modal responses of many modes can be measured simultaneously and complex mode shapes can be directly identified instead of relying upon and being constrained by the so called "normal mode" concept. Furthermore, the entire system response matrix, which comprises the mass, stiffness, and damping matrices of the lumped equivalent model, can be measured.

The following material covers the theoretical background that is needed to understand these new measurement techniques.

COMPLEX MODES AND THE TRANSFER MATRIX

Let's assume that the motion of a linear physical system can be described by a set of n simultaneous second order linear differential equations in the time domain, given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \qquad (1)$$

where the dots denote differentiation with respect to time. **f = f(t)** is the applied force vector, and **x = x(t)** is the resulting displacement vector, while **M**, **C**, and **K** are the (n by n) mass, damping, and stiffness matrices respectively. In this discussion, our attention will be limited to symmetric matrices, and to real element values in **M**, **C**, and **K**.

Taking the Laplace transform of the system equations gives

$$\mathbf{B}(\mathbf{s})\mathbf{X}(\mathbf{s}) = \mathbf{F}(\mathbf{s}), \text{ where} \qquad (2)$$

$$\mathbf{B}(\mathbf{s}) = \mathbf{M}\mathbf{s}^2 + \mathbf{C}\mathbf{s} + \mathbf{K} \qquad (3)$$

Here, **s** is the Laplace variable, and now **F(s)** is the applied force vector and **X(s)** is the resulting displacement vector in the Laplace domain. **B(s)** is called the system matrix, and the transfer matrix H(s) is defined as

$$\mathbf{H}(\mathbf{s}) = \mathbf{B}(\mathbf{s})^{-1} \qquad (4)$$

which implies that

$$\mathbf{H}(\mathbf{s})\mathbf{F}(\mathbf{s}) = \mathbf{X}(\mathbf{s}) \qquad (5)$$

Each element of the transfer matrix is a transfer function. The elements of **B** are quadratic functions of **s**, and since $\mathbf{H} = \mathbf{B}^{-1}$, it follows that the elements of **H** are rational fractions in **s**, with det(**B**) as the denominator. Thus, **H(s)** can always be represented in partial fraction form.

If it is assumed that the poles of **H**, i.e. the roots of det(**B**) = 0, are of unit multiplicity, then **H** can be expressed as

$$\mathbf{H} = \sum_{\mathbf{k}=1}^{\mathbf{2n}} \frac{\mathbf{a_k}}{\mathbf{s} - \mathbf{s_k}} \qquad \text{(n by n)} \qquad (6)$$

The poles occur at $\mathbf{s} = \mathbf{s_k}$ (zeros of det **B**), and each pole has an (n x n) residue matrix $\mathbf{a_k}$ describing its spatial behavior. For an n^{th} order oscillatory system, there will always be 2n poles, but they will appear in complex conjugate pairs. The poles are complex numbers expressed as

$$\mathbf{s_k} = \boldsymbol{\sigma}_\mathbf{k} + \mathbf{i}\boldsymbol{\omega}_\mathbf{k} \quad (7)$$

where $\boldsymbol{\sigma}_\mathbf{k}$ is the damping coefficient (a negative number for stable systems), and $\boldsymbol{\omega}_\mathbf{k}$ is the natural frequency of oscillation. The resonant frequency is given by

$$\boldsymbol{\omega}_\mathbf{rk} = \sqrt{\boldsymbol{\sigma}_\mathbf{k}^2 + \boldsymbol{\omega}_\mathbf{k}^2} \qquad \text{rad/sec} \quad (8)$$

and the damping factor is

$$\varsigma = -\frac{\boldsymbol{\sigma}_\mathbf{k}}{\boldsymbol{\omega}_\mathbf{rk}} \quad (9)$$

In reference [1], modal vectors were derived in terms of the eigenvectors of the system matrix **B**. However, these eigenvectors were only introduced as an intermediary in the determination of modal vectors. Here, the modal vectors are described in terms of the **B** matrix. Pre-multiplying **B** times the expression for **H**, multiplying by the scalar $(\mathbf{s} - \mathbf{s_k})$, and letting $\mathbf{s} = \mathbf{s_k}$ gives

$$\mathbf{B_k a_k} = 0, \text{ where } \mathbf{B_k} = \mathbf{B(s_k)} \quad (10)$$

Similarly, post-multiplication of **H** by **B** gives

$$\mathbf{a_k B_k} = 0 \quad (11)$$

Thus, *all rows and columns* of $\mathbf{a_k}$ must comprise linear combination of homogeneous solution vectors $\mathbf{u_k}$, given by

$$\mathbf{B_k u_k} = 0 \quad (12)$$

These $\mathbf{u_k}$ homogeneous solution vectors are defined as modal (or mode shape) vectors associated with the pole at $\mathbf{s} = \mathbf{s_k}$.

Restricting our attention to the special case where only one $\mathbf{u_k}$ modal vector exists for each pole, it is clear that all rows and columns of $\mathbf{a_k}$ must be some scalar multiple of $\mathbf{u_k}$. Thus, $\mathbf{a_k}$ can be represented by

$$\mathbf{a_k} = \mathbf{A_k u_k u_k}^\mathbf{t} \qquad \text{(n by n)} \quad (13)$$

where $\mathbf{A_k}$ is a scalar. In these terms, **H** can be rewritten as

$$\mathbf{H} = \sum_{\mathbf{k=1}}^{\mathbf{2n}} \left(\frac{\mathbf{A_k}}{\mathbf{s} - \mathbf{s_k}} \right) \mathbf{u_k u_k}^\mathbf{t} \quad (14)$$

and this is easily written in matrix form as

$$\mathbf{H} = \boldsymbol{\theta}\boldsymbol{\lambda}^{-1}\boldsymbol{\theta}^\mathbf{t} \qquad \text{(n by n)} \quad (15)$$

where the columns of $\boldsymbol{\theta}$ comprise the $\mathbf{u_k}$ modal vectors:

$$\boldsymbol{\theta} = \begin{bmatrix} | & | & & & | \\ \mathbf{u_1} & \mathbf{u_2} & \bullet & \bullet & \mathbf{u_{2n}} \\ | & | & & & | \end{bmatrix} \qquad \text{(n by 2n)} \quad (16)$$

and $\boldsymbol{\lambda}^{-1}$ is a diagonal matrix containing all **s** dependence:

$$\boldsymbol{\lambda}^{-1} = \begin{bmatrix} \frac{\mathbf{A_1}}{\mathbf{s} - \mathbf{s_1}} & & & & \mathbf{0} \\ & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \\ \mathbf{0} & & & & \frac{\mathbf{A_{2n}}}{\mathbf{s} - \mathbf{s_{2n}}} \end{bmatrix}, \text{(2n x 2n)} \quad (17)$$

Pre-multiplying **H** by $\boldsymbol{\theta}^\mathbf{t}$, equation (5) can be written as

$$\left(\boldsymbol{\theta}^\mathbf{t}\boldsymbol{\theta}\boldsymbol{\lambda}^{-1}\right)\left(\boldsymbol{\theta}^\mathbf{t}\mathbf{F}\right) = \left(\boldsymbol{\theta}^\mathbf{t}\mathbf{X}\right) \quad (18)$$

so that $\boldsymbol{\theta}^\mathbf{t}$ transforms the spatial vectors (**F** and **X)** to vectors $\boldsymbol{\theta}^\mathbf{t}\mathbf{F}$ and $\boldsymbol{\theta}^\mathbf{t}\mathbf{X}$ in modal coordinates. Similarly $\left(\boldsymbol{\theta}^\mathbf{t}\boldsymbol{\theta}\boldsymbol{\lambda}^{-1}\right)$' is the modal representation of **H**.

Since $\mathbf{B(s_k)u_k} = 0$ it follows that $\mathbf{B(s_k^*)u_k^*} = 0$, so the modal vector associated with the conjugate pole ($\mathbf{s_k}^*$) is ($\mathbf{u_k}^*$), (the conjugate of $\mathbf{u_k}$). Thus, the above $\boldsymbol{\theta}$ matrix always contains conjugate pairs of modal vectors and the $\boldsymbol{\lambda}^{-1}$ matrix always contains elements corresponding to conjugate pole pairs along its diagonal. If poles are purely imaginary (zero damping) then all modal vectors are real, so only half of $\boldsymbol{\theta}$ is needed. Otherwise $\boldsymbol{\theta}$ is rectangular (n x 2n) and consequently, even though **H** can be written in diagonal form using the $\boldsymbol{\theta}$ matrix, its inverse **B** *cannot be diagonalized* using $\boldsymbol{\theta}$ except in the special case when damping is zero.

IDENTIFICATION OF MODAL PARAMETERS

Because of the form of the $\mathbf{a_k}$ matrix, *only one row or column* of the transfer matrix need be measured and analyzed, since all other rows and columns contain redundant information. In the process of measuring the transfer matrix, unknown parameters in equation (14), i.e. the complex values of $\mathbf{s_k}$ and the complex values of the elements of one row or column of $\mathbf{a_k}$ are identified.

For example, the q^th^ column of $\mathbf{a_k}$ is given by

$$\mathbf{a_{kq}} = \mathbf{A_k u_k u_{kq}} = \mathbf{A_k u_{kq} u_k} \tag{19}$$

where $\mathbf{u_{kq}}$ is the q^th^ element of $\mathbf{u_k}$. Thus the modal vector $\mathbf{u_k}$ (whose normalization is arbitrary) can be recovered once the q^th^ column of $\mathbf{a_k}$ is identified. In addition the complex scalar $\mathbf{A_k}$ can be recovered using the formula

$$\mathbf{A_k} = \frac{\sqrt{\mathbf{a_{kq}}^{\mathbf{t}}\mathbf{a_{kq}}}}{\mathbf{u_{kq}}\sqrt{\mathbf{u_k}^{\mathbf{t}}\mathbf{u_k}}} \tag{20}$$

It is clear that the numerical value of $\mathbf{A_k}$ depends upon the normalization of the modal vector. If we choose $\mathbf{u_k}^{\mathbf{t}}\mathbf{u_k} = 1$, then

$$\mathbf{A_k} = \frac{1}{\mathbf{u_{kq}}}\sqrt{\mathbf{a_{kq}}^{\mathbf{t}}\mathbf{a_{kq}}} \tag{21}$$

Since complex modal vectors appear in conjugate pairs, **H** can always be written in two parts as

$$\mathbf{H} = \sum_{\mathbf{k=1}}^{\mathbf{n}}\left[\frac{\mathbf{A_k}}{\mathbf{s}-\mathbf{s_k}}\mathbf{u_k u_k}^{\mathbf{t}} + \frac{\mathbf{A_k}^*}{\mathbf{s}-\mathbf{s_k}^*}\mathbf{u_k}^*\mathbf{u_k}^{*\mathbf{t}}\right] \tag{22}$$

For the k^th^ pair of conjugate poles,

$$\mathbf{H_k} = \frac{\mathbf{A_k}}{\mathbf{s}-\mathbf{s_k}}\mathbf{u_k u_k}^{\mathbf{t}} + \frac{\mathbf{A_k}^*}{\mathbf{s}-\mathbf{s_k}^*}\mathbf{u_k}^*\mathbf{u_k}^{*\mathbf{t}} \tag{23}$$

Each pole of the transfer matrix has a corresponding mode shape vector and, furthermore, each complex conjugate pair of poles has a corresponding complex conjugate <u>pair</u> of mode shapes. The pq^th^ element of $\mathbf{H_k}$ is

$$\mathbf{H_{kpq}} = \frac{\mathbf{A_k}}{\mathbf{s}-\mathbf{s_k}}\mathbf{u_{kp} u_{kq}} + \frac{\mathbf{A_k}^*}{\mathbf{s}-\mathbf{s_k}^*}\mathbf{u_{kp}}^*\mathbf{u_{kq}}^* \tag{24}$$

The time domain displacement at point p due to an impulsive force at point q is given by the inverse Laplace transform of $\mathbf{H_{kpq}}$ which is

$$\begin{aligned}\mathbf{h_{kpq}(t)} &= \mathbf{A_k u_{kp} u_{kq} e^{s_k t}} + \mathbf{A_k}^*\mathbf{u_{kp}}^*\mathbf{u_{kq}}^*\mathbf{e}^{\mathbf{s_k}^*\mathbf{t}} \\ &= \mathbf{2e^{\sigma_k t}}\left[\mathbf{Re}\left(\mathbf{A_k u_{kp} u_{kq}}\right)\mathbf{cos(\omega_k t)}\right. \\ &\quad \left.- \mathbf{Im}\left(\mathbf{A_k u_{kp} u_{kq}}\right)\mathbf{sin(\omega_k t)}\right] \\ &= \mathbf{2e^{\sigma_k t}}\left|\mathbf{A_k u_{kp} u_{kq}}\right|\mathbf{cos(\omega_k t + \alpha_k)}\end{aligned} \tag{25}$$

for $\mathbf{t \geq 0}$.

Note that the peak amplitude of the impulse response is $\left|\mathbf{A_k u_{kp} u_{kq}}\right|$ and the phase angle $(\alpha_{\mathbf{k}})$ is the angle of the complex residue $\mathbf{A_k u_{kp} u_{kq}}$ with respect a cosine. The magnitude and phase of this complex residue are different, in general, for each spatial point on the structure.

MASS, STIFFNESS, AND DAMPING MATRICES

Recall that $\mathbf{B}(\mathbf{s}) = \mathbf{Ms}^2 + \mathbf{Cs} + \mathbf{K}$, so it is apparent that

$$\mathbf{K} = \mathbf{B}(0) = \mathbf{H}(0)^{-1} \tag{26}$$

Now, **H(0)** is obtained by setting **s = 0** in λ^{-1}.

A modal compliance κ^{-1} can be defined as

$$\kappa^{-1} = \lambda^{-1}(0) = \begin{bmatrix} -\frac{\mathbf{A}_1}{\mathbf{s}_1} & & & & 0 \\ & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \\ 0 & & & & -\frac{\mathbf{A_{2n}}}{\mathbf{s_{2n}}} \end{bmatrix}$$

(2n x 2n) (27)

So

$$\mathbf{K} = \mathbf{H(0)^{-1}} = \left(\theta\kappa^{-1}\theta^{\mathbf{t}}\right)^{-1} \tag{28}$$

Thus, the stiffness matrix is readily obtained from the measured modal vectors (columns of θ), and the identified $\mathbf{A_k}$ and $\mathbf{s_k}$ complex scalars. Since

$$\mathbf{HB} = \mathbf{I} \tag{29}$$

it follows that

$$\mathbf{HB'} + \mathbf{H'B} = \mathbf{0} \qquad (30)$$

and

$$\mathbf{HB''} + 2\mathbf{H'B'} + \mathbf{H''B} = 0 \qquad (31)$$

where the prime denotes differentiation with respect to **s**.

The damping matrix can be computed with the following relationship

$$\mathbf{C} = \mathbf{B'}(0) = -\mathbf{B}(0)\mathbf{H'}(0)\mathbf{B}(0) = -\mathbf{KH'}(0)\mathbf{K} \qquad (32)$$

Or alternatively by defining the modal damping matrix δ as

$$\delta = \begin{bmatrix} \frac{1}{\mathbf{A}_1} & & & & 0 \\ & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \\ 0 & & & & \frac{1}{\mathbf{A}_{2\mathbf{n}}} \end{bmatrix} \qquad (2n \times 2n) \qquad (33)$$

then

$$\mathbf{H'}(0) = \theta\left(\lambda^{-1}\right)'(0)\theta^{\mathbf{t}} = -\theta\kappa^{-1}\delta\kappa^{-1}\theta^{\mathbf{t}} \qquad (34)$$

and

$$\mathbf{C} = \left(\mathbf{K}\theta\kappa^{-1}\right)\delta\left(\mathbf{K}\theta\kappa^{-1}\right)^{\mathbf{t}} \qquad (35)$$

In a similar manner, we can write

$$\begin{aligned} \mathbf{M} &= \frac{1}{2}\mathbf{B''}(0) \\ &= -\mathbf{B}(0)\mathbf{H'}(0)\mathbf{B'}(0) - \frac{1}{2}\mathbf{B}(0)\mathbf{H''}(0)\mathbf{B}(0) \\ &= -\mathbf{KH'}(0)\mathbf{C} - \frac{1}{2}\mathbf{KH''}(0)\mathbf{K} \end{aligned} \qquad (36)$$

But **H"(0)** can be written as

$$\mathbf{H''}(0) = 2\theta\kappa^{-1}\delta\kappa^{-1}\delta\kappa^{-1}\theta^{\mathbf{t}} \qquad (37)$$

Finally, the mass matrix can be obtained from

$$\begin{aligned} \mathbf{M} &= \left(\mathbf{K}\theta\kappa^{-1}\right)\left(-\delta\kappa^{-1}\delta\right)\left(\mathbf{K}\theta\kappa^{-1}\right)^{\mathbf{t}} \\ &\quad + \left(\mathbf{K}\theta\kappa^{-1}\right)\delta\kappa^{-1}\theta^{\mathbf{t}}\mathbf{C} \\ &= \left(\mathbf{K}\theta\kappa^{-1}\right)\left(-\delta\kappa^{-1}\delta\right)\left(\mathbf{K}\theta\kappa^{-1}\right)^{\mathbf{t}} + \mathbf{CK}^{-1}\mathbf{C} \end{aligned} \qquad (38)$$

Now, from the expression for **K**, we obtain

$$\mathbf{K}\theta\kappa^{-1} = \left(\theta\kappa^{-1}\theta^{\mathbf{t}}\right)^{-1}\theta\kappa^{-1} = \theta_0^{-\mathbf{t}} \qquad (39)$$

where $\theta^{-\mathbf{t}}$ is the left-handed inverse of $\theta^{\mathbf{t}}$ defined by

$$\theta^{-\mathbf{t}}\theta^{\mathbf{t}} = \mathbf{I} \qquad (40)$$

To summarize, the stiffness, damping, and mass matrices are obtained from the expressions

$$\mathbf{K} = \left(\theta\kappa^{-1}\theta^{\mathbf{t}}\right)^{-1} = \theta_0^{-\mathbf{t}}\kappa\theta_0^{-1} \qquad (41)$$

$$\mathbf{C} = \theta_0^{-\mathbf{t}}\delta\theta_0^{-1} \qquad (42)$$

$$\mathbf{M} = \theta_0^{-\mathbf{t}}\mu\theta_0^{-1} + \mathbf{CK}^{-1}\mathbf{C} \qquad (43)$$

where

$$\mu = -\delta\kappa^{-1}\delta \qquad \text{(diagonal)} \qquad (44)$$

All of these quantities are readily obtained from the measured poles $\mathbf{s_k}$, scalars $\mathbf{A_k}$, and modal vectors $\mathbf{u_k}$.

COMMENTARY

The fundamental nature of complex modes in the real world of measurement cannot be over-emphasized. Damping is always present in a structure, and it can always be observed in measured transfer function data. The fact that a complex mode implies a complex time waveform should be of little concern, because the conjugate waveform is always present to make the observed signal real valued. A complex mode has the character of a "traveling wave" across the structure (as opposed to the usual "standing wave" produced by a normal mode) as indicated by the changing phase angle of the displacement from point to point.

Furthermore, it is important to note that a mode shape still a global (as opposed to local) property of a structure , even though damping may be heavy. The fact that local motion near the point of excitation in a heavily damped structure tends to dominate is generally caused by many closely spaced modes that are excited in phase at this point, but tend to cancel each other elsewhere.

Historically, considerable emphasis has been placed on simultaneously diagonalizing the mass and stiffness matrices, so that the displacement of a particular point could be easily calculated from an arbitrary excitation force. Unfortunately, it is not possible to simultaneously diagonalize more than two symmetric matrices, so this technique cannot be used when damping is present.

However, when a system is represented by the partial fraction form of its transfer matrix, a closed form solution for the displacement at any point, for any combination of modes, is readily obtainable using simple matrix-vector multiplication. This is particularly helpful when the response to only a few modes is of interest.

The definition of a modal vector as a solution to the homogeneous system equation is also very fundamental, and removes much of the ambiguity about what modal vectors really are, and when they exist. This definition also makes them relatively easy to calculate or measure.

It should be apparent from the equations for obtaining mass, damping, and stiffness from measured data, that the order of these matrices is equal to the number of complex modal pairs that are measured. This means that if **n** modes are identified, only **n** spatial points are needed to represent the lumped equivalent model of the physical system. This result can be stated as the following theorem.

> **SPATIAL SAMPLING THEOREM:** A lumped system modeled with second order elements has exactly **2n** poles and **2n** modal vectors. Thus, if only **n** modal pairs are found, there is no need for more than **n** spatial points in the lumped model.

Finally, the scope of this discussion has been limited to symmetric matrices having distinct poles (unit multiplicity) in complex conjugate pairs. Even though these assumptions are satisfied by a large majority of the linear systems tested, modes of vibration can still be defined when all of these assumptions are relaxed, and we have worked out the theory for this general case.

SUMMARY

We have defined the complex modal (mode shape) vector associated with each pole (zero of the system determinant) of an elastic mechanical system as the homogeneous solution to the system equations. We have also emphasized that these poles occur in complex conjugate pairs, and that the two associated modal vectors are also conjugates of one another.

The transfer matrix (inverse of system matrix) was introduced, and it was written in partial fraction form, comprising one term for each pole. We found that the rows and columns of the residue matrix associated with each pole are multiples of the corresponding modal vector. The modal matrix θ , whose columns are the modal vectors, was defined, and it was shown that θ^t will transform a vector from spatial coordinates to modal coordinates.

Finally, we derived expressions for mass stiffness, and damping in terms of the poles and modal vectors, and indicated how measured parameters can be used to calculate these matrices.

REFERENCES

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