MEASUREMENT AND ANALYSIS OF THE DYNAMICS OF MECHANICAL STRUCTURES

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INTRODUCTION

Nowadays, government regulations, consumer action groups, oil crises, inflation, and many other factors are putting pressure on the automotive industry, and many other industries, to build products which weigh less and operate more reliably and efficiently, with less noise and vibration than in the past. To design products which meet these more stringent requirements, design engineers must have better testing and analysis tools to assist them.

This paper discusses modern methods for testing, modeling and analyzing the dynamics of mechanical structures. The topics covered are listed in Figure 1. As indicated, much of the discussion concerns the modes of vibration of mechanical structures which turn out to be the link between the testing and analysis methods discussed in this paper.

Mechanical design problems can be divided into four categories, depending upon whether forces or motion on the structure are taken into account. These four alternatives are shown in the matrix in Figure 2.

When forces and motion on a structure are ignored, the design problem can be considered to be one of styling. Styling, in this respect, can include determining the sizes and shapes of components, and the entire layout of the mechanical system, not just its outward appearance.

If the structure must withstand or transmit certain known forces, then a static analysis may be carried out to insure that certain components of the structure are strong enough to withstand the forces. For example, the suspension springs in an automobile must have sufficient strength to support to weight of the body and engine, etc.

In most cases, any rotating or moving parts of a mechanical system must be laid out so that they move with the proper motion with respect to one another. These design problems are known as kinematics problems. For example in an automobile engine, there are numerous rotating gears, shafts, pulleys, chains, cams, etc. that must undergo the proper motion relative to one another in order for the engine to run.

A designer may do a thorough job of analyzing the effects of forces and motion individually on a structure, but once it is subjected to a dynamic environment, where forces and motion occur together, he finds that it has many noise and vibration problems. These dynamics problems, which are influenced by inertial forces and the elastic behavior of the structure, are often very difficult to solve.

MODELING STRUCTURAL DYNAMICS

For the purposes of modeling their dynamics, mechanical structures can be divided into two classes; RIGID bodies and ELASTIC bodies. Although most structures are elastic throughout, many times analysis of their dynamics can be greatly simplified without significant loss of accuracy by assuming that they behave as if they were an assemblage of rigid masses connected together by spring and damper elements. (A simple mass, spring, damper system is shown in Figure 3.) Rigid body systems (also called lumped parameter systems) can be analyzed in a straightforward manner by applying Newton's Second

TOPICS

1. MODELING STRUCTURAL DYNAMICS
2. MODES OF VIBRATION—THE LINK BETWEEN ANALYSIS AND MEASUREMENTS
3. REVIEW OF MODAL ANALYSIS
4. THE TRANSFER FUNCTION TECHNIQUE FOR STRUCTURAL TESTING
5. IDENTIFYING MODAL PARAMETERS FROM TRANSFER FUNCTION MEASUREMENTS

FIGURE 1

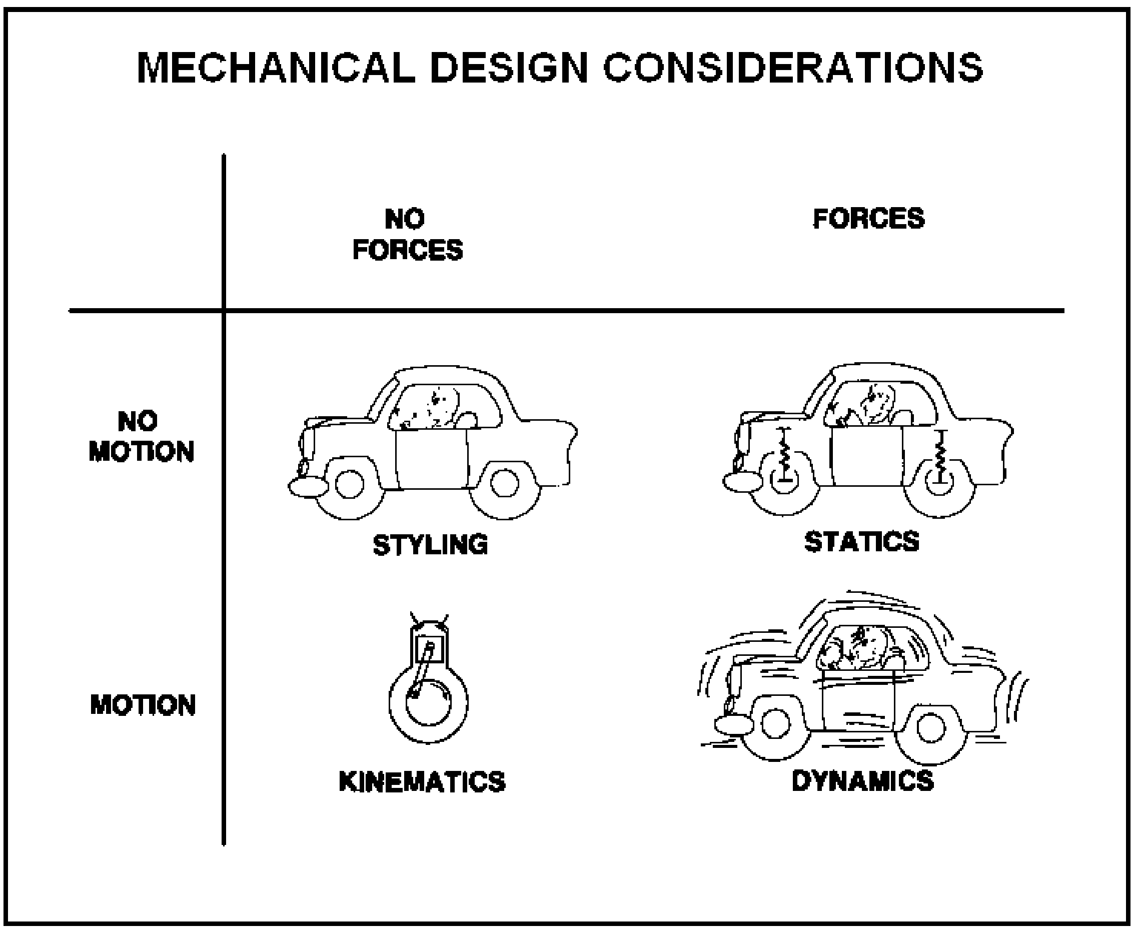


FIGURE 2

Law to all masses in the system. By performing a "force balance" on the system, that is by setting all forces acting on each mass equal to its mass multiplied by its acceleration, a set of differential

equations is derived which completely describes the dynamics of the system. These equations can then be solved analytically to yield structural responses to specified external forces and boundary conditions.

However many dynamics problems cannot be solved by using a rigid body approximation of the structure. These problems require that the distributed elastic behavior of the structure be modeled more accurately.

If the structure has a simple geometric shape and its physical properties (eg. density and elasticity) are more or less uniform throughout, then a partial differential equation of the form known as the "wave equation" can be used to describe its dynamics. There are well known solutions to the wave equation for many types of simple mechanical structures, such as beams, shafts and plates. However the approximations required in order to apply these analytical methods are often too restrictive to adequately describe the dynamics of a complex structure such as automobile body.

The requirement for a more generalized method for modeling the dynamics of large, complex structures with non-homogeneous physical properties has brought about the development in recent years of the finite element modeling method.

The Finite Element Method

The objective of the finite element method is to sub-divide a structure into an assemblage of many smaller elements such as plates, beams, shafts, etc. Then the overall equations of motion of the structure are constructed from equations describing the motions of each of the individual elements, plus all the boundary conditions at the connection points between elements. (This process is depicted in Figure 4.)

A primary advantage of this approach is that it has been computerized, and readily available programs such as the NASTRAN program, which was initially developed by NASA (The National Aeronautics and Space Administration of the United States) to assist in the design of space vehicles, can be used to build very large dynamic models of complex structures.

As shown in Figure 4, the differential equations of motion are a set of simultaneous, second order differential equations which represent a "force balance" among the inertial, dissipative, restoring and externally applied forces on the structure.

Note that the space variable $x(t)$ has been "discretized" in this modeling process. That is, a finite number of degrees-of-freedom have been chosen to represent the motion of the structure. (A degree-of-freedom is motion at a point in a particular direction.) This approximation removes all derivatives with respect to the space variable from the equations and reduces them to a "generalized" statement of Newton's Second Law.

Although these equations are of the same form as those obtained by using rigid body analysis, the mass, stiffness, and damping coefficient matrices do not necessarily contain mass values, spring constants and damping coefficients which are readily associated with lumped elements on the structure. Rather, these matrices can be viewed simply as coefficients which are necessary to satisfy force balances between the various finite elements of the structure.

Once the mathematical model has been built, (i.e. the mass, stiffness and damping matrices have been synthesized), the equations of motion can be solved, again by using computer methods.

A popular method of solving the equations of motion is to "diagonalize" them. This is done by finding the "eigenvalues" and "eigenvectors" of the equations. A commonly used approach is to assume that the

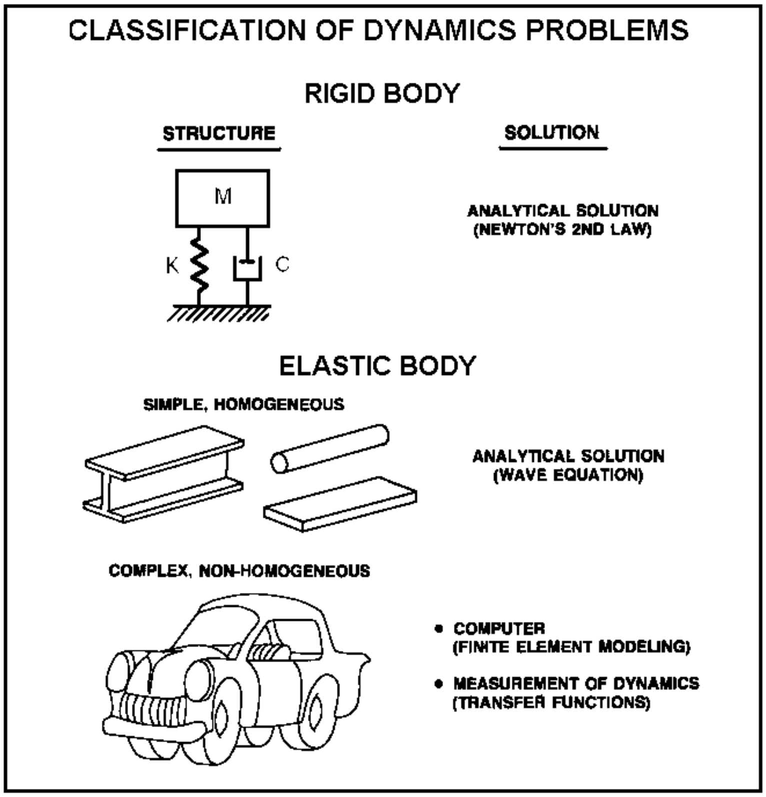
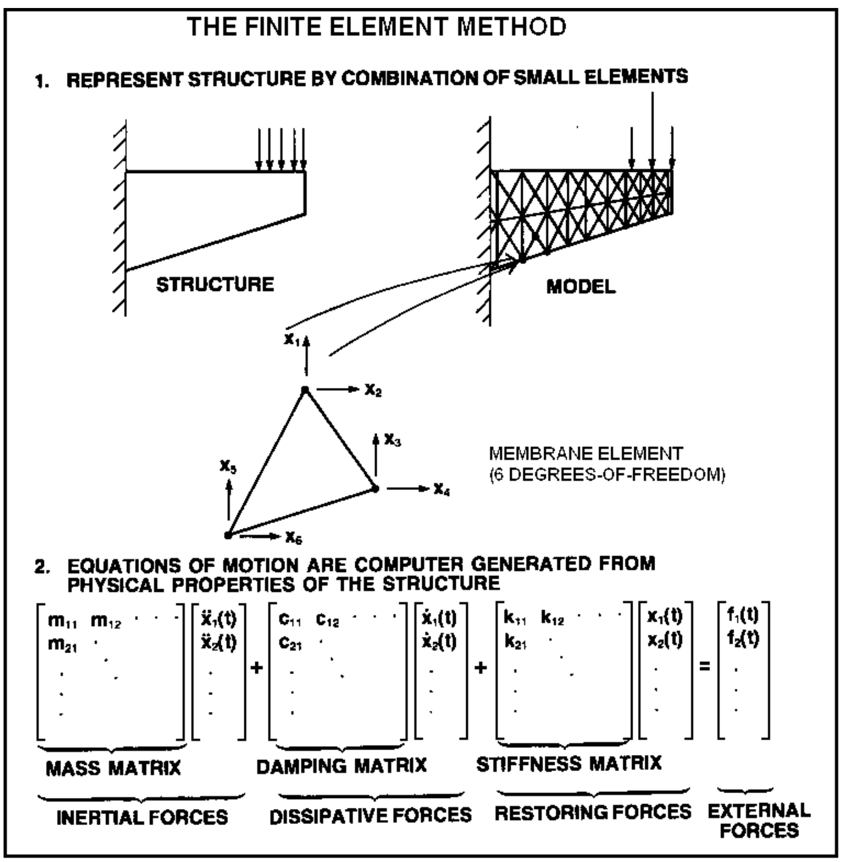


FIGURE 3



$$\underbrace{\begin{bmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & \ddots & \\ \vdots & & \end{bmatrix}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \vdots \end{bmatrix}}_{\text{INERTIAL FORCES}} + \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & \ddots & \\ \vdots & & \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \end{bmatrix}}_{\text{DISSIPATIVE FORCES}} + \underbrace{\begin{bmatrix} k_{11} & k_{12} & \cdots \\ k_{21} & \ddots & \\ \vdots & & \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}}_{\text{RESTORING FORCES}} = \underbrace{\begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \end{bmatrix}}_{\text{EXTERNAL FORCES}}$$

FIGURE 4

THE FINITE ELEMENT METHOD

3. THE EQUATIONS OF MOTION ARE "DIAGONALIZED" SO THAT RESPONSES TO EXTERNAL FORCES CAN BE SIMULATED
   - DIAGONALIZATION INVOLVES FINDING THE "EIGENVALUES" AND "EIGENVECTORS" OF THE EQUATIONS OF MOTION
   - DISSIPATIVE (DAMPING) FORCES ARE USUALLY IGNORED

$$\underbrace{\begin{bmatrix} m_1 & o & \cdots \\ o & m_2 & \\ \vdots & & \ddots \end{bmatrix}}_{\text{GENERALIZED MASSES}} \underbrace{\begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ \vdots \end{bmatrix}}_{\text{GENERALIZED COORDINATE ACCELERATION}} + \underbrace{\begin{bmatrix} k_1 & o & \cdots \\ o & k_2 & \\ \vdots & & \ddots \end{bmatrix}}_{\text{GENERALIZED STIFFNESSES}} \underbrace{\begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \end{bmatrix}}_{\text{GENERALIZED COORDINATES}} = \underbrace{\begin{bmatrix} u_{11} & u_{21} & \cdots \\ u_{12} & \ddots & \\ \vdots & & \ddots \end{bmatrix}}_{\text{EIGENVECTOR MATRIX}} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \cdots \\ u_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \end{bmatrix}$$

FIGURE 5

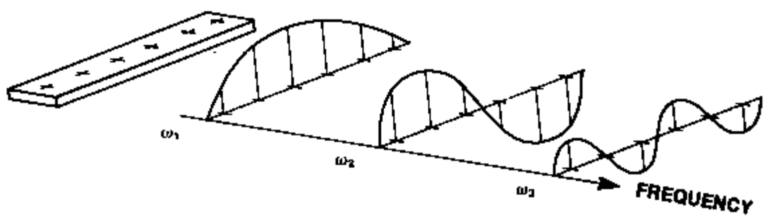
USES OF A MATHEMETICAL MODEL

1. LOADS ANALYSIS: EXTERNAL FORCES ⇨ INTERNAL STRESSES, STRAINS

2. MODAL ANALYSIS: IDENTIFICATION OF RESONANT FREQUENCIES OF VIBRATION AND AMPLITUDES OF VIBRATION AT RESONANT FREQUENCIES

   EIGENVALUES ⇨ RESONANT FREQUENCIES

   EIGENVECTORS ⇨ AMPLITUDES OF VIBRATION (MODAL VECTORS, MODE SHAPES)



3. "WHAT IF" INVESTIGATIONS: WHAT HAPPENS IF MASS AND/ OR STIFFNESS IS ADDED TO OR REMOVED FROM STRUCTURE (MODAL SYNTHESIS)

4. DYNAMIC SIMULATION: STRUCTURE RESPONSE TO REAL WORLD EXTERNAL FORCES (FATIGUE PREDICTION)

FIGURE 6

damping forces on the structure are negligible, and can therefore be ignored. The equations of motion, minus the damping terms, can be transformed to a new coordinate system, called "generalized" coordinates, and written in diagonal or uncoupled form as shown in Figure 5. The transformation relating the Generalized coordinates to the actual degrees-of-freedom of the structure is a matrix, the columns of which are the eigenvectors of the system. Once the equations of motion are in diagonal form it is much easier to solve them to obtain the structure's response to externally applied forces. This model is known as a "normal mode" model.

Uses of the Mathematical Model

A finite element model can be used to perform several different types of analysis. They include: LOAD ANALYSIS, MODAL ANALYSIS, WHAT IF INVESTIGATIONS and DYNAMIC SIMULATION.

LOAD ANALYSIS is basically an analysis of the internal stresses and strains in a structure due to external loads. If the structure is in a static condition (i.e. all accelerations are zero) then the equations of motion reduce to a force balance between the internal restoring forces and externally applied forces. By performing analyses with these equations, areas of high static stress or strain can be located on the structure. In a similar manner dynamic stress and strain levels can be analyzed if the inertial terms are also included in the equations of motion.

MODAL ANALYSIS is defined as the process of characterizing the dynamics of a structure in terms of its modes of vibration. It turns out that the eigenvalues and eigenvectors of the previously defined normal mode mathematical model are also parameters which define the resonant frequencies and mode shapes of the modes of vibration of the structure. (This is shown in Figure 6.) That is, the eigenvalues of the equations of motion correspond to frequencies at which the structure tends to vibrate with a predominant, well defined deformation. The amplitude of this wave motion on the structure is specified by the corresponding eigenvector. Each mode of vibration, then, is defined by an eigenvalue (resonant frequency) and corresponding eigenvector (mode shape). If the dynamic model has n-degrees-of-freedom then it also has n-eigenvalue-eigenvector pairs, or n-modes of vibration.

Knowing the modes of vibration of a structure is useful information in itself, for it tells at what frequencies the structure can be excited into resonant motion, and the predominant wave-like motion it will assume at a resonant frequency. In many cases, this information is sufficient for modifying the structural design in order to reduce noise and vibration.

WHAT IF INVESTIGATIONS can be conducted using a finite element model to determine how changes in the mass or stiffness of the structure will affect its dynamic characteristics. These types of investigations can be made using the mathematical model long before the first prototype structure is even built. This way, 4D any deficiencies in the design can be spotted early in the design cycle where changes are a lot less costly than in the later stages. This capability is perhaps the single most important advantage of finite element modeling.

A finite element model can also be used for SIMULATION of the dynamic response of the structure to real world external forces. These forces might be of short duration and high amplitude, i.e. impulsive in nature, in which case they could cause immediate damage to the structure. Or they may be of long duration and cyclic in nature, and could cause fatigue damage to the structure over a long period of time.

The simulated response of the mathematical model could be used as input to fatigue damage prediction algorithms, thus giving information about the fatigue life of a prototype design even before the first prototype is constructed.

FINITE ELEMENT MODELING

ADVANTAGES:

1. MATH MODEL CAN BE BUILT BEFORE PROTOTYPE
2. "WHAT IF" INVESTIGATIONS ARE CHEAPER, FASTER AND EASIER THAN ALTERING REAL HARDWARE

DISADVANTAGES:

1. LARGE MODELS ARE VERY EXPENSIVE TO RUN
2. MODEL CAN BE INACCURATE

FIGURE 7

WHY DYNAMIC (MODAL) TESTING?

1. TO CONFIRM ANALYTICALLY-DERIVED DYNAMIC MODELS
2. TO TROUBLESHOOT VIBRATION PROBLEMS
3. TO EVALUATE DESIGN "FIXES" TO STRUCTURES
4. TO CONSTRUCT A DYNAMIC MODEL FOR PARTS OF A STRUCTURE TOO DIFFICULT TO MODEL ANALYTICALLY

FIGURE 8

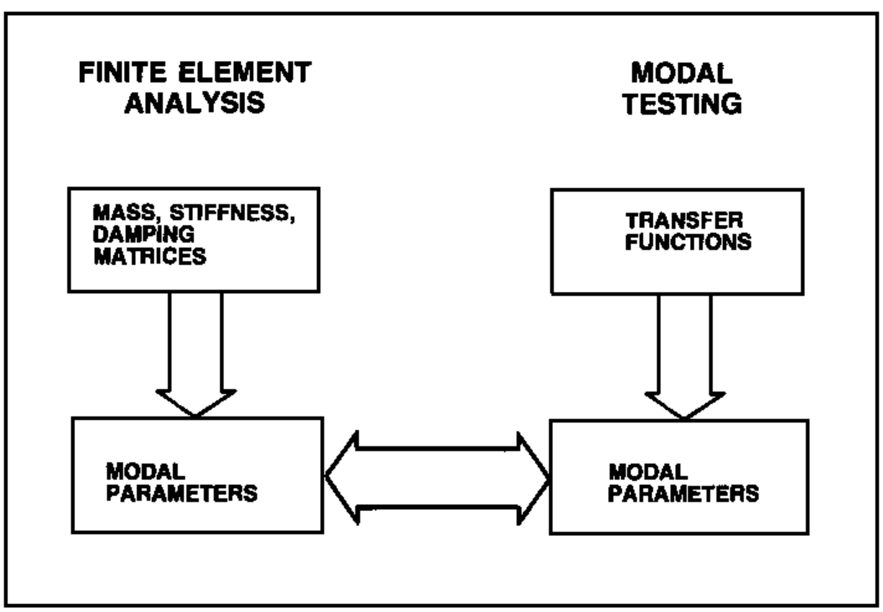


FIGURE 9

Finite element modeling does however have two major disadvantages, as pointed out in Figure 7. Most finite element computer programs are very large in size, and require large computers with lots of memory in which to operate. Hence, it is not unusual for companies to spend tens of thousands of dollars to develop a single finite element model. To obtain the required accuracy, models containing several thousand degrees-of-freedom are not uncommon. Models of this size require many man-hours of effort to develop, debug, and operate.

A second disadvantage of finite element modeling is that the dynamic response of the model can differ substantially from that of the actual structure. This can occur because of errors in entering model parameters, but can also occur when the finite elements do not approximate the real world situation well enough. Many times the model will turn out to be much stiffer than the actual structure. This can be due to the use of an inadequate number of elements or unrealistic boundary conditions between elements. Both of these disadvantages point to a need for dynamic testing of the structure in order to confirm the validity of the model.

Why Dynamic Testing?

Not only is dynamic testing necessary in order to check a finite element model, but some other Advantages can be gained from testing, as shown in Figure 8.

Dynamic testing can be used for troubleshooting noise and vibration problems in existing mechanical systems. These problems can occur because of errors in the design or construction of the system, or they could occur as a result or wearout, failure or malfunction in some of its components. Not only can testing be used to locate a problem, but it can also be used to evaluate fixes to the problem. Finally dynamic testing can be used to construct a dynamic model for components of a structure which are too difficult to model analytically.

In all the cases mentioned above, the objective of the dynamic testing procedure is to excite and identify the test specimen's modes of vibration. As shown in Figure 9 the common element between finite element modeling and dynamic, (or modal) testing is the modal parameters of the structure.

MODES OF VIBRATION

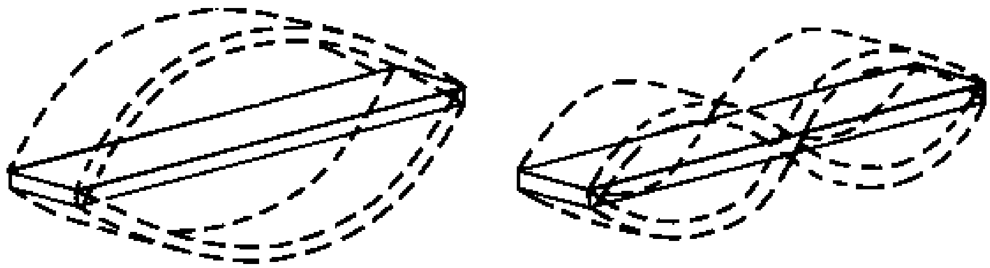
As previously discussed, one of the mathematical transformations which can be performed on the dynamic model of a structure converts it to an "uncoupled" form. This process involves identifying the eigenvalues and eigenvectors of the equations of motion. These parameters also define the modes of vibration of the structure.

Not only are modal parameters a fundamental part of a mathematical dynamic model, but they can be observed in practically any vibrating body, Physically speaking modes of vibration are the so called "natural" frequencies at which a structure's predominant motion is a well defined waveform, as shown in Figure 10.

Mathematically speaking modes of vibration are defined by certain parameters of a linear dynamic model. As shown in Figure 11 each mode of vibration is defined by a resonant frequency, a damping factor and a mode shape. It will be shown later that a dynamic model can be completely represented in terms of these parameters.

The purpose of modal testing, then, is to artificially excite a structure so that the frequencies, damping and mode shapes of its predominant modes of vibration can be identified.

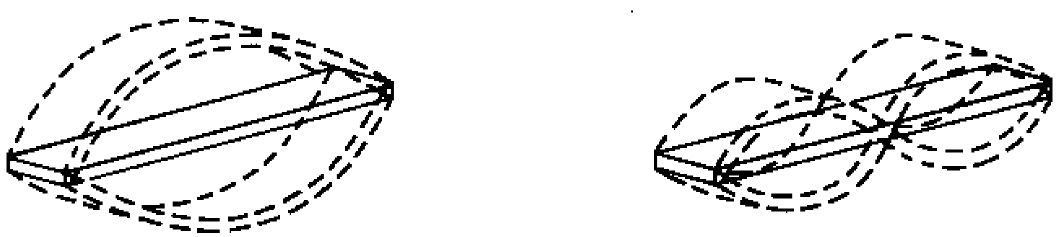
MODES OF VIBRATION (PHYSICAL): "NATURAL" FREQUENCIES AT WHICH A STRUCTURE'S PREDOMINANT MOTION IS A WELL DEFINED WAVEFORM



MODES OF VIBRATION (MATHEMATICAL): PARAMETERS OF A LINEAR DYNAMIC MODEL

FIGURE 10

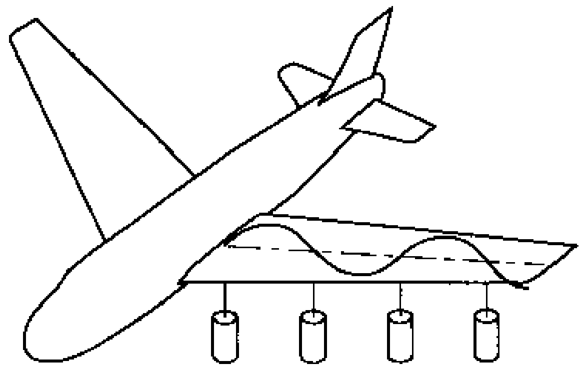
EACH MODE IS DEFINED BY



MODE 1		MODE 2
23.58 Hz	RESONANT FREQUENCY	45.75 Hz
1.54%	DAMPING FACTOR	2.04%
(1st BENDING)	MODE SHAPE	(2nd BENDING)

FIGURE 11

NORMAL MODE METHOD



- WIDE BAND SWEEP
- NARROW BAND SWEEP (FREQUENCY)
- TUNING AND MODAL DWELL (MODE SHAPE)
- DECAY MEASUREMENTS (DAMPING)

FIGURE 12

TRANSFER FUNCTION METHOD

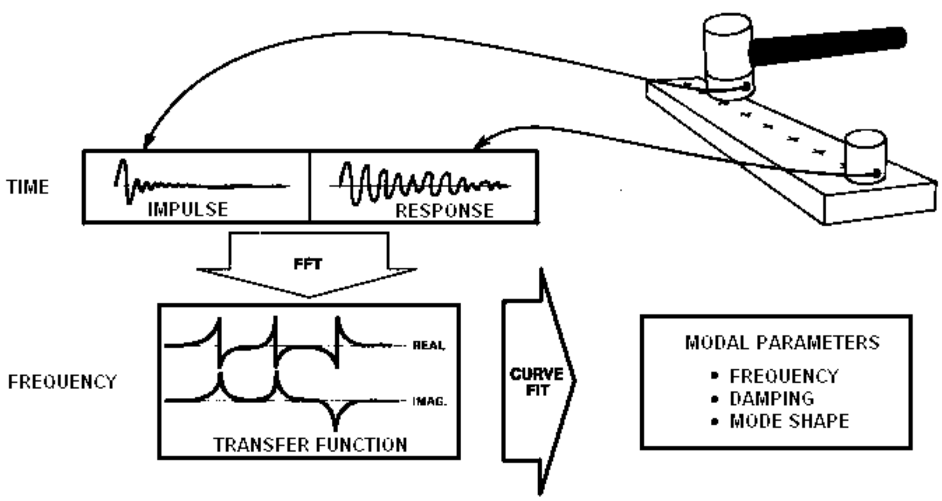
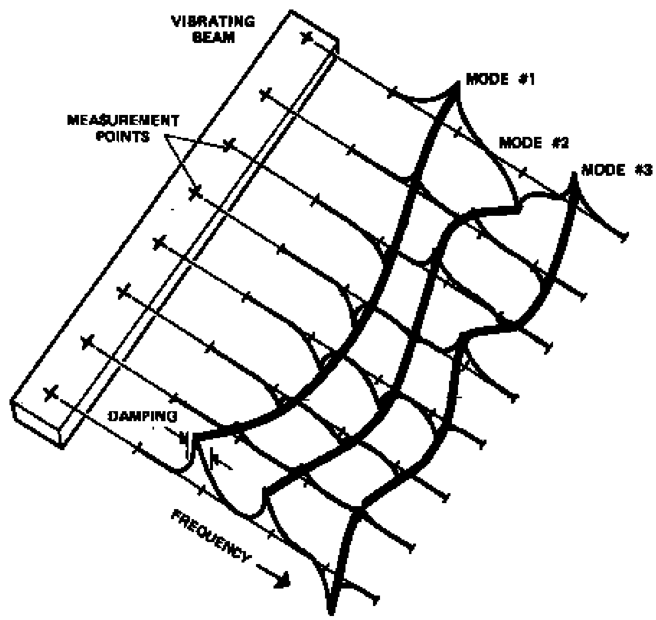


FIGURE 13

MODAL DATA FROM TRANSFER FUNCTIONS



DAMPING & FREQUENCY — SAME AT EACH MEASUREMENT POINT

MODE SHAPE— OBTAINED AT SAME FREQUENCY FROM ALL MEASUREMENT POINTS

FIGURE 14

MODAL TESTING METHODS

NORMAL MODE METHOD	TRANSFER FUNCTION METHOD
MULTI-SHAKER	SINGLE POINT OR MULTI-SHAKER
SINUSOIDAL EXCITATION	BROADBAND EXCITATION
ONE MODE AT A TIME	MANY MODES AT A TIME
ANALOG INSTRUMENTATION	FFT-BASED DIGITAL INSTRUMENTATION

FIGURE 15

MODAL TESTING METHODS

Two fundamentally different methods of modal testing are used today for testing structures. These methods are referred to as the NORMAL MODE method and the TRANSFER FUNCTION method. The NORMAL MODE method is the more traditional of the two, and has been used for the past 25 years by the aerospace industry to test large spacecraft and aircraft structures. The TRANSFER FUNCTION method has become popular within the past 5 to 10 years and is being used by many manufacturing industries, including the automotive and machine tool industries.

The Normal Mode Method

The primary objective of this method is to excite the undamped (or "normal") modes of a structure, one at a time. This is typically done by attaching several shakers to the structure, as shown in Figure 12, and driving them with a sinusoidal signal equal in frequency to the natural frequency of the mode to be excited. The amplitudes and polarity of the sinusoidal drive signals are adjusted so that the externally applied forces so that the predominant motion of the structure is due to the desired mode of vibration.

Resonant frequencies of modes are first located by performing so called WIDE BAND sweeps, usually with only one shaker active. once a rise in the amplitude of response is detected (by watching the response signal on an oscilloscope), a NARROW BAND sweep is performed using smaller frequency changes to more accurately identify the frequency of the mode.

Once a resonant condition is located, multiple shakers are turned on in an effort to excite a single mode of vibration.

The process of adjusting the amplitude, polarity, and frequency of the shakers to excite a normal mode is called MODAL TUNING. Once a mode is properly excited its amplitudes of vibration at many points on the structure are measured, and taken as the mode shape. This condition is referred to as MODAL DWELL.

Then, to measure the damping of the mode, the shakers are simultaneously shut off to simulate an impulse response of the structure at the frequency of the mode. Ideally the structure should exhibit a damped sinusoidal response at all points, with a single frequency of vibration being the frequency of the excited mode. Typically if more than one mode was excited during modal tuning and dwell, the impulse responses will show a "beating" of several modal frequencies.

If, however, a so-called "pure" mode was excited, the damping of the structure at the modal frequency can be measured from the envelope of the damped sinusoidal response.

There are a number of problems which make this testing method difficult, time consuming, and expensive to implement. First of all, it is difficult to know where to locate shakers on the structure without some forehand knowledge of the modes of vibration. Secondly, it is often extremely difficult to excite closely coupled modes (i.e. close in frequency with heavy damping) one at a time. Thirdly, since all the mode shape data is collected during modal dwell, the structure must be completely instrumented with enough transducers and signal conditioning equipment so that amplitudes for all the desired degrees-of-freedom can be measured at once.

The Transfer Function Method

This method has gained much popularity in recent years because it is faster and easier to perform, and is much cheaper to implement than the normal mode testing method.

The major steps of the transfer function method are depicted in Figure 13. It is based upon the use of digital signal processing techniques and the FFT (Fast Fourier Transform) algorithm to measure transfer functions between various points on the structure. For example, on the simple beam in Figure 13, a set of transfer functions is measured between each of the X's marked on the beam, and a single response point. A single transfer function measurement is obtained by exciting the beam with a hammer at one of the X's, simultaneously measuring the input force and corresponding response motion signals, and then dividing the Fourier transform of the response by the transform of the input.

Modal parameters are identified by performing further computations (i.e. "curve fitting") on this set of transfer function measurements. Figure 14 shows how modal parameters can be obtained from transfer function measurements. Although much more sophisticated curve fitting algorithms are often used to identify modal parameters, this figure shows fundamentally how the parameters are obtained. The figure shows the imaginary part of each transfer function made between an impact point and the reference point. Modal frequencies correspond to peaks in the imaginary part of the transfer functions. A peak should exist at the same frequency in all measurements, except those measured at "node" points where the modal amplitude is zero. The width of the modal peak is related to the damping of the mode. That is, the wider the peak, the higher the modal damping. The mode shape is obtained by assembling the peak values at the same frequency from all measurements. As shown in Figure 14, as modal frequency increases, the complexity of the mode shape also increases.

Some major differences between the two modal testing methods are shown in Figure 15. A fundamental difference is that one method attempts to excite one mode at a time using a narrow band signal, while the other attempts to excite many modes simultaneously, using a broadband signal. A distinct advantage of the transfer function method is that any type of broadband excitation method can be used since the measurement being made is a response signal divided by the input which caused it. This type of "normalized" response measurement is independent of the type of input signal used, as long as it can be measured and has sufficient energy to excite the structure over the frequency range of interest. Hence, a simple excitation device such as a hammer can many times be used to measure transfer functions at a great savings of time and money compared to attaching shakers to the structure.

Another cost advantage on the transfer function method is that the measurements can be made one at a time. This means that a large set of measurements can be made using only one accelerometer (motion transducer) and one load cell (force transducer), and the corresponding signal conditioning equipment. If a shaker is used for excitation, the accelerometer is moved to each new measurement point on the structure. If a hammer is used, the structure is impacted at each new measurement point.

Some other advantages of the transfer function method are shown in Figure 16. In general, it is easier to make transfer function measurements on a structure than to isolate one of its modes of vibration. In addition, once the measurement signals have been digitized and stored in the computer's memory, further processing of the data can be performed to reduce the effects of noise and distortion. Large amounts of data can also be stored on a mass memory device such as a magnetic disc or tape, and later recalled for further processing. Statistical estimation algorithms which use large amounts of measurement data, can also be used to estimate modal parameters with more accuracy.

Lastly, the mode shapes can be displayed in animation on the same equipment used to perform the modal test. Figures 17 and 18 depict a typical modal test setup using the transfer function method, and some typical displays of mode shapes of the test structure on the CRT of a transfer function analyzer. Not only is the mode shape display an effective method for checking the validity of a large amount of data, (i.e. data values in gross error are easily spotted) but the animation is also a convenient means of locating "weak spots" or areas of overdesign in the structure.

In addition to these direct uses of modal data for problem solving purposes, it is next shown that modal parameters are a fundamental part of a transfer function dynamic model of a structure.

ADVANTAGES OF THE TRANSFER FUNCTION METHOD

1. EASIER TO MAKE MEASUREMENTS
   - IMPACT TESTING IS QUICK & INEXPENSIVE
   - PRIOR KNOWLEDGE OF MODES NOT REQUIRED
2. DIGITAL ACCURACY AND REPEATABILITY
3. ESSENTIALLY UNLIMITED FREQUENCY RESOLUTION
4. REDUCED EFFECTS OF NOISE AND NONLINEAR DISTORTION
5. STATISTICAL ESTIMATION OF MODAL PARAMETERS
6. ANIMATED MODE SHAPE

FIGURE 16

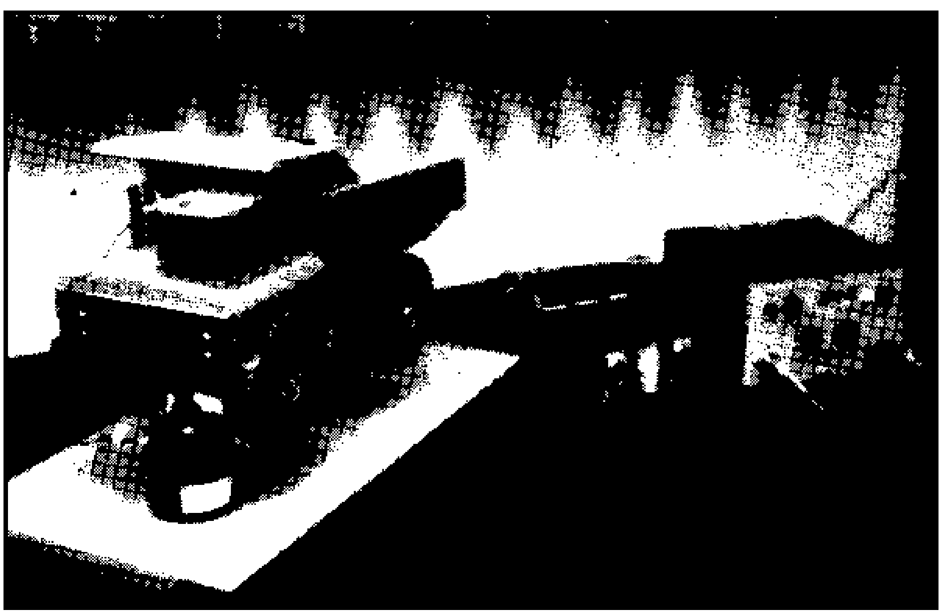


FIGURE 17

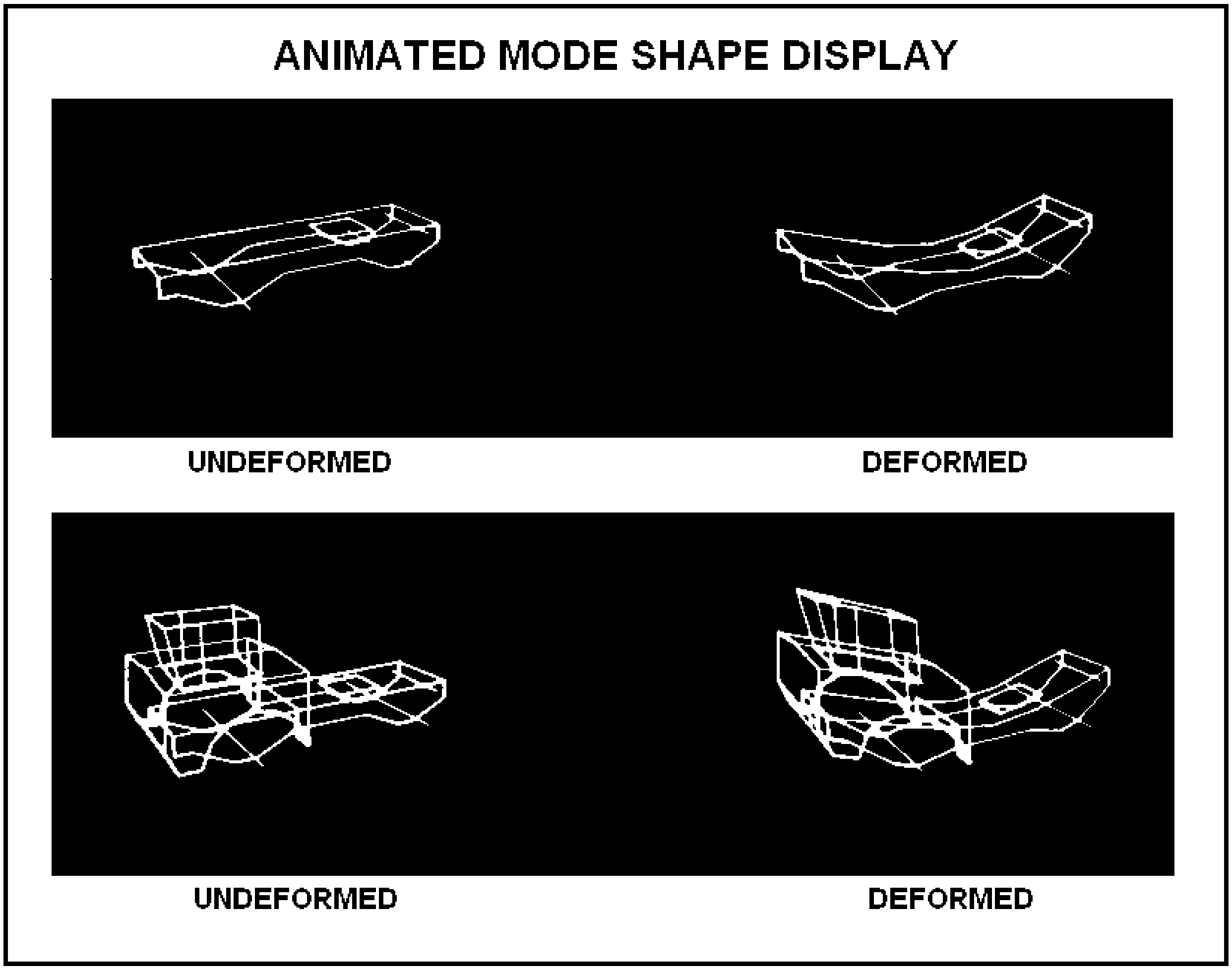

ANIMATED MODE SHAPE DISPLAY
UNDEFORMED
DEFORMED
UNDEFORMED
DEFORMED

FIGURE 18

REVIEW OF MODAL ANALYSIS USING TRANSFER FUNCTIONS

Modes of vibration have already been defined in terms of the eigenvalues and eigenvectors of the time domain equations of motion. Since the transfer function method of modal testing is based upon the measurement of frequency domain functions, it is next shown that modes of vibration can be defined by parameters of a transfer function matrix model of the structure, which is equivalent to the time domain model.

It should become clear from this analysis that during a typical modal test, one row or column of the transfer matrix model is being measured, and that this is sufficient information to identify all the parameters which define the modes of vibration. It should also become clear that a complete dynamic model can be constructed from the modal parameters.

Time Domain Model

In a measurement situation the actual input forces and responses for a finite number of degrees-of-freedom of the structure are measured. S 0 if a model were constructed from the measurements involving these specific degrees-of-freedom, the model would give an accurate description of the structural dynamics involving those points. This is a different situation than with a finite element model where the degrees-of-freedom and the size and shape of the elements are chosen so as to approximate the dynamics of the structure as closely as possible.

The time domain structural dynamic model, as shown in Figure 19, exhibits the same form as the finite element model but, at least in principle, is an exact model of the structural dynamics if obtained from measurements.

Lap1ace Domain Model.

We do not directly measure the time domain model of Figure 19, but rather its Laplace domain equivalent, shown in Figure 20.

In this model the inputs and responses of the structure are represented by their Laplace transforms. Time domain derivatives (i.e. velocity and acceleration) do not appear explicitly in the Laplace domain model but are accounted for in the transfer functions. The transfer matrix contains transfer functions which describe the effect of an input at each degree-of-freedom (D.O.F.) upon the response at each D.O.F. Because the model is linear, the transformed total motion for any D.O.F. is the sum of each transformed input force multiplied by the transfer function between the input D.O.F. and the response D.O.F.

For example

$$\mathbf{X}_1(\mathbf{s}) = \mathbf{h}_{11}(\mathbf{s})\mathbf{F}_1(\mathbf{s}) + \mathbf{h}_{12}(\mathbf{s})\mathbf{F}_2(\mathbf{s}) + \ldots + \mathbf{h}_{1\mathbf{n}}(\mathbf{s})\mathbf{F}_{\mathbf{n}}(\mathbf{s})$$

Transfer Function of a Single Degree-of-Freedom

The Laplace variable is a complex number, normally denoted by $\mathbf{s = \sigma + j\omega}$. Since the transfer function is a function of the s-variable, it too is complex valued. Plots of a typical transfer function on the S-plane are shown in Figure 22. Because it is complex, the transfer function is represented by its REAL and IMAGINARY parts or equivalently by its MAGNITUDE and PHASE. Note that in this case, the transfer function is only plotted over half of the S-plane, i.e. it is not plotted for any positive values of $\boldsymbol{\sigma}$. This was done to give a clear picture of the transfer function values along the $\mathbf{j\omega}$-axis. These values will become important later in this development.

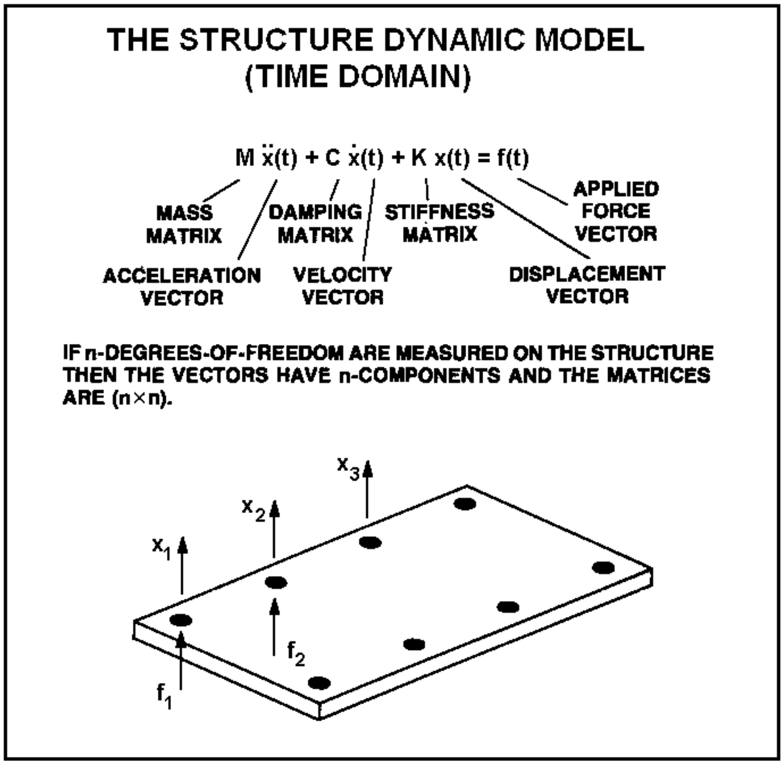


FIGURE 19

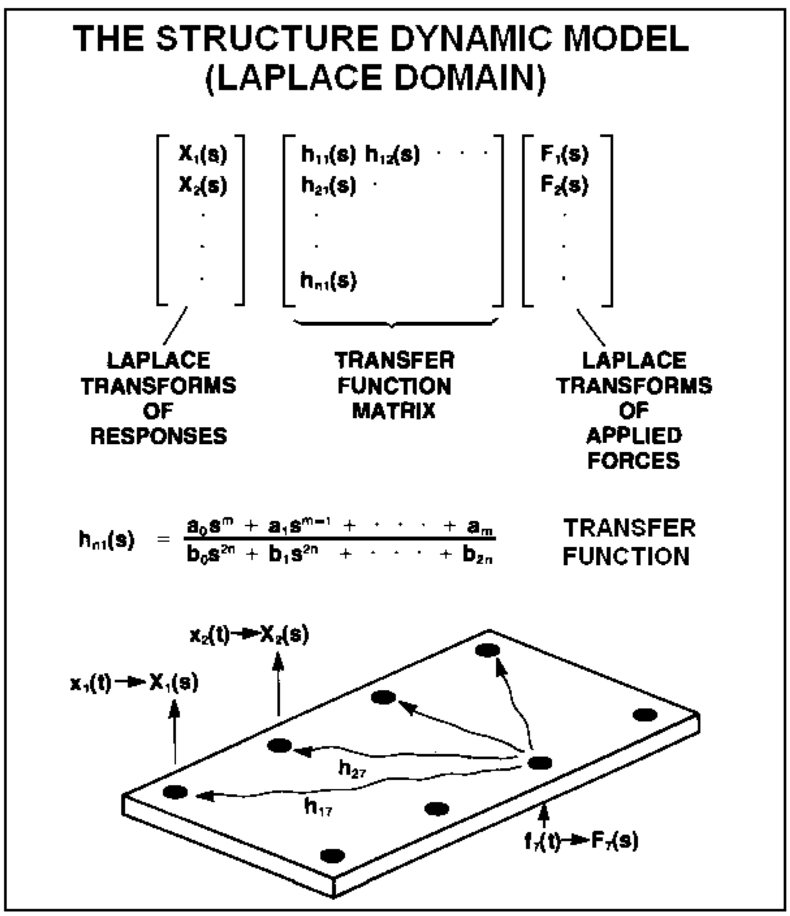


FIGURE 20

Note also that the magnitude of the transfer function goes to infinity at two points in the S-plane. These discontinuities are called the POLES of the transfer function. These poles define resonant conditions on the structure which will "amplify" an input force. The location of these poles in the S-plane is defined by a FREQUENCY and DAMPING value as shown in Figure 23. Hence the $\boldsymbol{\sigma}$-axis and $\mathbf{j}\boldsymbol{\omega}$-axis of the S-plane have become known as the damping axis and the frequency axis respectively. The frequency and damping which define a pole in the S-plane are the frequency and damping of a mode of vibration of the structure.

Transfer Matrix in Partial Fraction Form

The elements of the transfer matrix can be written as ratios of polynomials as shown in Figure 20. With some minor assumptions, (explained later) the transfer matrix-can be re-written in partial fraction form as shown in Figure 21. This form clearly shows the transfer function in terms of the parameters which describe its pole locations, namely $\mathbf{p_k} = \boldsymbol{\sigma_k} + \mathbf{j}\boldsymbol{\omega_k}$. For a model with n-degrees-of-freedom, it is clear that the transfer functions contain n-pole pairs $(\mathbf{p_k}, \mathbf{p_k^*})$.

(* - denotes complex conjugate.) Two unique features of the partial fraction form are that all transfer functions contain the same denominator terms involving the S-variable, and that the numerators simply become constants (numbers), which are assembled into the RESIDUE matrix, and its conjugate matrix.

Transfer Matrix in Terms of Modal Parameters

After writing it in partial fraction form, the transfer function can be further simplified by writing the residue matrix in terms of a MODAL VECTOR (u_k) as shown in Figure 24.

The derivation of this form is given in Refs. (1) or (2). This is a crucial step, for now we have reduced the transfer matrix (i.e. the entire dynamic model) to a parametric form involving only modal parameters. As stated in Figure 24 a mode of vibration is characterized by a pair of conjugate poles and a pair of conjugate mode vectors.

Note that the unique form of the residue matrix allows the entire (n x n) matrix to be defined once the n-dimensional mode vector is known. Furthermore since every row and column contains the mode vector multiplied by a different component of itself, ONLY ONE ROW OR COLUMN of the residue matrix (and hence the transfer matrix) needs to be measured in order to identify the mode vector. A 2-dimensional case is written out in Figure 25 to illustrate this point.

The numerators of the first column of the transfer matrix are made up of the mode vector (u_{11}, u_{21}) multiplied by its first component (u_{11}), plus the conjugate mode vector (u_{11}^*, u_{21}^*)

multiplied by its first component (u_{11}^*), for mode #1. Similarly, two more terms are added for mode #2. The denominators, which contain the pole locations (p_1,p_1^*) and (p_2, p_2^*), are the same for every transfer function in the matrix.

TRANSFER FUNCTION MATRIX IN PARTIAL FRACTION FORM

$$H(s) = \sum_{k=1}^{n} \left[\frac{r_k}{s - p_k} + \frac{r_k^*}{s - p_k^*} \right]$$

$p_k = \sigma_k + j\,\omega_k$ $\quad$ k^{th} POLE

r_k – MATRIX OF RESIDUES FOR k^{th} POLE

$$r_k = \begin{bmatrix} r_{11}(k) & r_{12}(k) & \cdots \\ r_{21}(k) & \ddots & \\ \vdots & & \end{bmatrix} \quad (n \times n)$$

FIGURE 21

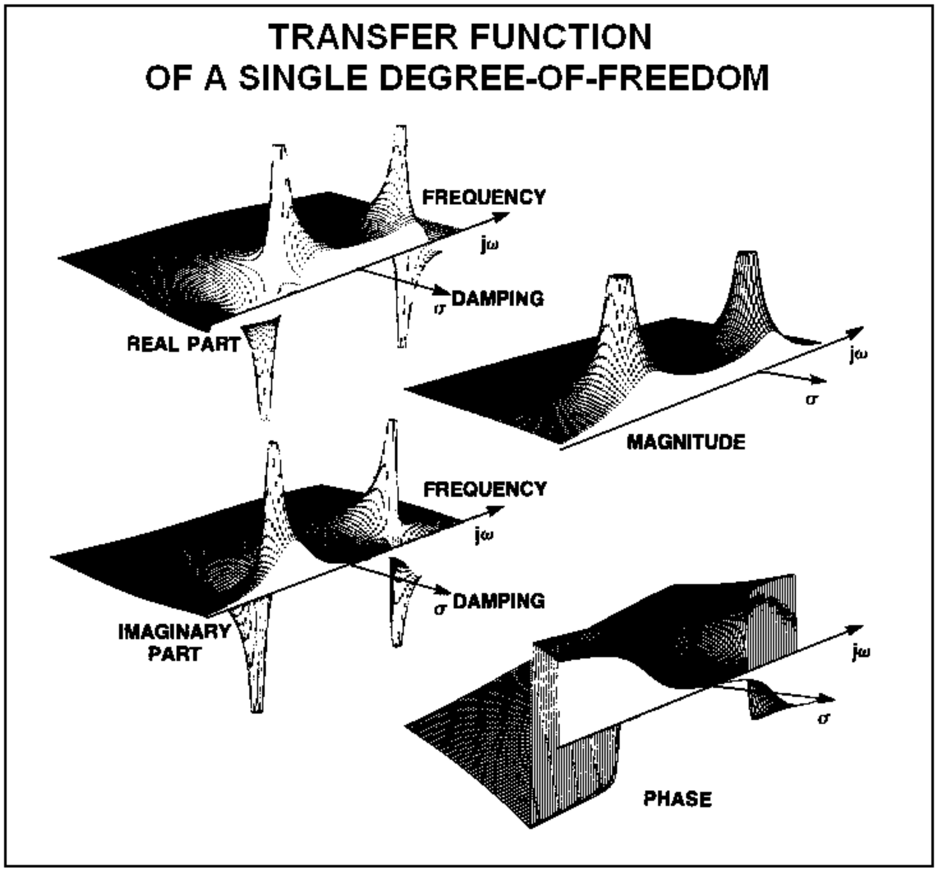

TRANSFER FUNCTION
OF A SINGLE DEGREE-OF-FREEDOM
FREQUENCY
jω
σ
DAMPING
REAL PART
jω
σ
MAGNITUDE
FREQUENCY
jω
σ
DAMPING
IMAGINARY PART
jω
σ
PHASE

FIGURE 22

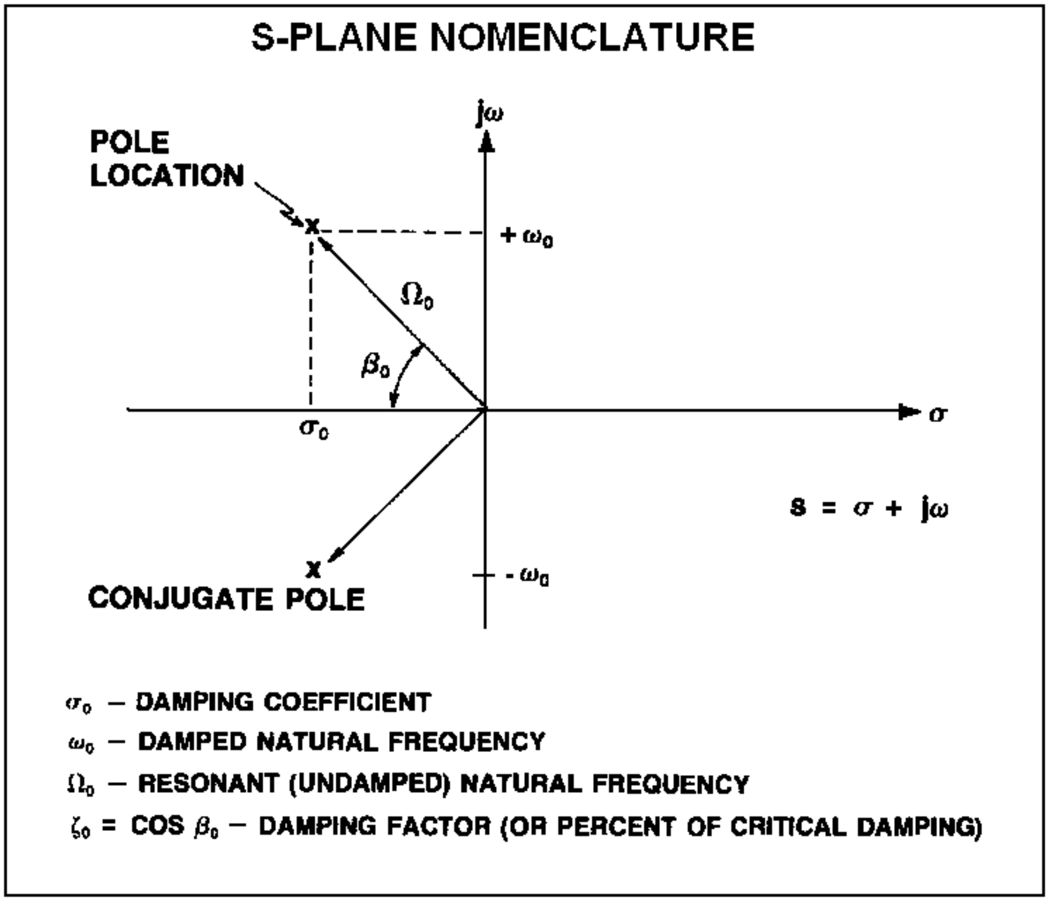

S-PLANE NOMENCLATURE
POLE LOCATION
$j\omega$
$+\omega_0$
Ω_0
β_0
σ_0
σ
$s = \sigma + j\omega$
$-\omega_0$
CONJUGATE POLE
σ_0 — DAMPING COEFFICIENT
ω_0 — DAMPED NATURAL FREQUENCY
Ω_0 — RESONANT (UNDAMPED) NATURAL FREQUENCY
ζ_0 = COS β_0 — DAMPING FACTOR (OR PERCENT OF CRITICAL DAMPING)

FIGURE 23

TRANSFER MATRIX IN TERMS OF MODAL PARAMETERS

$$H(s) = \sum_{k=1}^{n} \left[\frac{u_k u_k^{t}}{s - p_k} + \frac{u_k^{*} u_k^{*t}}{s - p_k^{*}} \right]$$

u_k = k^{th} MODAL VECTOR (n – DIMENSIONAL)

MODE OF VIBRATION: A MODE (k) IS CHARACTERIZED BY A PAIR OF CONJUGATE POLES (p_k, p_k^*) AND A PAIR OF CONJUGATE MODAL VECTORS (u_k, u_k^*).

FIGURE 24

TWO DEGREE-OF-FREEDOM CASE

$$H(s) = \underbrace{\frac{1}{s-p_1}\left[\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}\begin{bmatrix} u_{11} & u_{21} \end{bmatrix}\right] + \frac{1}{s-p_1^*}\left[\begin{bmatrix} u_{11}^* \\ u_{21}^* \end{bmatrix}\begin{bmatrix} u_{11}^* & u_{21}^* \end{bmatrix}\right]}_{\text{MODE \#1}}$$

$$+$$

$$\underbrace{\frac{1}{s-p_2}\left[\begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}\begin{bmatrix} u_{12} & u_{22} \end{bmatrix}\right] + \frac{1}{s-p_2^*}\left[\begin{bmatrix} u_{12}^* \\ u_{22}^* \end{bmatrix}\begin{bmatrix} u_{12}^* & u_{22}^* \end{bmatrix}\right]}_{\text{MODE \#2}}$$

FIGURE 25

<u>Modal Testing Implications</u>

The modal testing implications of this final parametric form of the transfer function model are stated in Figure 26.

Normally only one row or column of the transfer matrix is measured in a modal test, although multiple row and column elements could be measured to obtain better estimates of the modal parameters. (Ref. (4) covers this subject).

The assumptions made in order to derive the parametric form of the model are explained in more detail in Reference (1). These assumptions can be satisfied in a large number of test situations but they must be kept in mind during a modal test since they can be easily violated when testing complex structures.

MODAL TESTING IMPLICATIONS

MODAL PARAMETERS CAN BE IDENTIFIED FROM
<u>ONE ROW OR ONE COLUMN</u>
OF THE TRANSFER FUNCTION MATRIX

ASSUMPTIONS

1. STRUCTURAL MOTION CAN BE DESCRIBED BY LINEAR SECOND ORDER EQUATIONS
2. THE STRUCTURE EXHIBITS SYMMETRY (OR RECIPROCITY)
3. ONLY ONE MODES EXISTS AT EACH POLE LOCATION
4. ALL MODES ARE GLOBAL (LOCAL MODES ARE GLOBAL WITH MODE SHAPES DEFINED IN LOCAL REGIONS)

FIGURE 26

TRANSFER FUNCTION OF A SINGLE DEGREE-OF-FREEDOM

- - - FREQUENCY RESPONSE FUNCTION

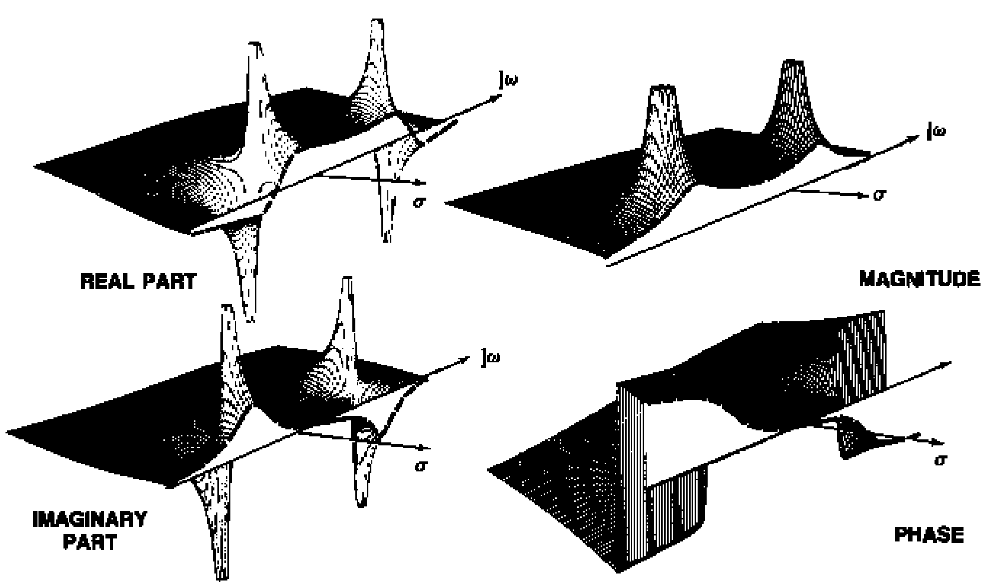


FIGURE 27

TRANSFER FUNCTION MEASUREMENT

In a test situation we do not actually measure the transfer function over the entire S-plane, but rather its values along the **jω-**axis. These values are known as the FREQUENCY RESPONSE FUNCTION, as shown in Figure 27. Since the transfer function is an "analytic" function, its values throughout the S-plane can be inferred from its values along the **jω**-axis. More specifically, if we can identify the unknown modal parameters of a transfer function by "curve fitting" the analytical form in Figure 21 to measured values of the function along the **jω**-axis, then we can synthesize the function throughout the S-plane.

Alternative Forms of the Frequency Response

The frequency response function, being complex valued, is represented by two numbers at each frequency. Figure 28 shows some of the alternative forms in which this function is commonly plotted. The so called CO-QUAD plot, or real and imaginary parts, derives its origin from the days of swept sine testing when the real part was referred to as the COincident waveform and the imaginary part as the QUADrature waveform. The Bode plot, or log magnitude and phase vs. frequency, is named after H.W. Bode who made many contributions to the analysis of frequency response functions. (Many of Bode's techniques involved plotting these functions along a log-frequency axis). The Nyquist plot, or real vs. imaginary part, is named after the gentleman who popularized its use for determining the stability of linear systems. The Nichols plot, or log magnitude vs. phase angle, is named after N.B. Nichols who used such plots to analyze servo-mechanisms.

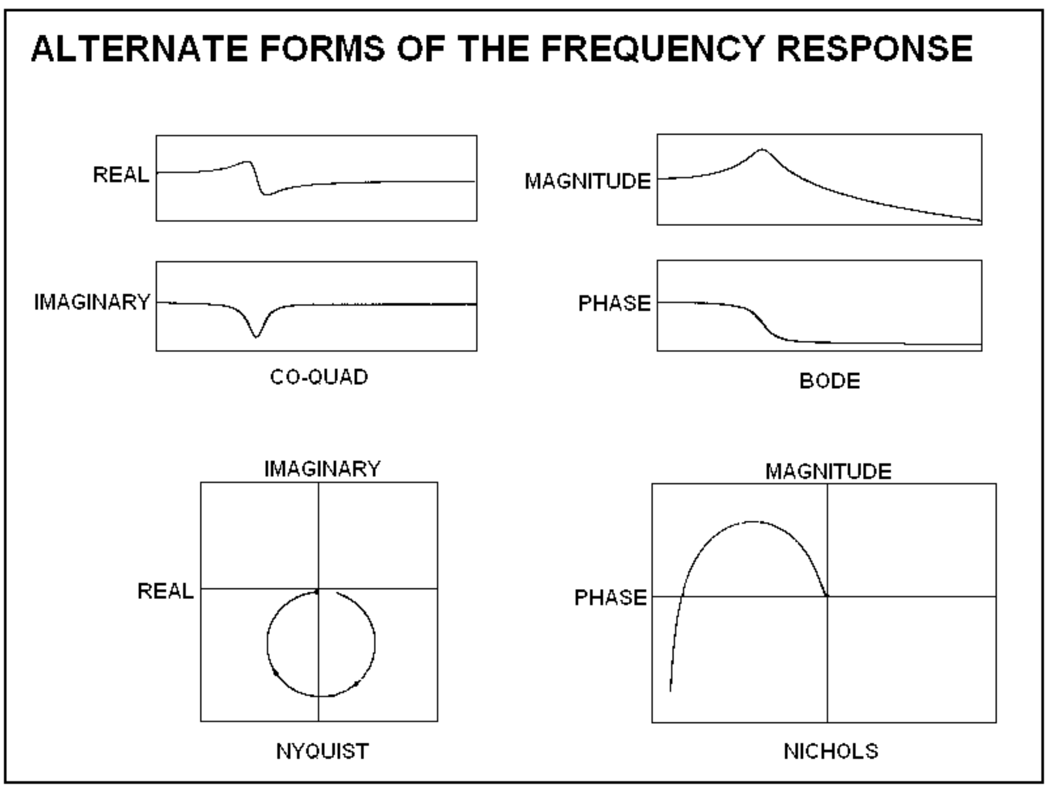


FIGURE 28

The Measurement Process

The transfer function measurement process is depicted in Figure 29. A key step in the measurement of the transfer function is that two signals from the structure, an input force signal and a response motion signal, are simultaneously digitized (sampled) and stored in the computer memory. From that point on, the measurement consists of performing mathematical operations on the digital data. The diagram in Figure 29 is actually a much simplified explanation of the signal processing that is typically used during transfer function measurements.

Measuring Elements of the Transfer Matrix

The simplest way of measuring elements of the transfer matrix (i.e. frequency response functions) is to measure them one at a time, as shown in Figure 30. In this simple 2-dimensional case the frequency response function ($h_{11}(\omega)$) is measured by exciting the structure at pt. #1 and measuring response at pt. #1. Then the function is formed by dividing the Fourier transform of the measured response motion ($X_1(\omega)$) by the Fourier transform of the measured input force ($F_1(\omega)$). Likewise the second element in the first row ($h_{12}(\omega)$ is measured by exciting the structure at pt. #2 and then dividing the Fourier transform of the response motion ($X_1(\omega)$) by the Fourier transform of the input force ($F_{21}(\omega)$). The second row of elements can be measured in a similar manner.

More sophisticated measurement methods involving multiple inputs and responses could be implemented, but this simplified single, input-single output approach is most commonly used.

If time savings is a significant test objective, as it often is in larger modal tests, than test time can be significantly reduced by measuring a single input force and several response motions simultaneously. From this data, several transfer functions in a single column of the transfer matrix can be computed.

Typical Measurement Setups

A typical measurement setup using a shaker driven by a broadband random signal is shown in Figure 31. This figure shows the measurement being made with a Fourier analyzer system. After the transducer signals are amplified, via charge amplifiers they are passed through low pass anti-aliasing filters before being digitized by the analog to digital converter (ADC) of the Fourier analyzer. Some transfer function analyzers contain built-in anti-aliasing filters, and some modern day transducers contain built-in charge amplifiers, which make them both more convenient to use.

The test specimen is normally mounted in a manner which allows it to vibrate freely (called a free-free condition), or in a manner which allows it to vibrate the way it would in an actual operating environment.

Figure 32 shows a typical test setup for transient testing. When a small hammer is used to excite the structure as shown in the example, then a set of measurements (i.e. a row of the transfix matrix) is normally obtained by impacting the structure at various points while measuring its response at a single stationary point. However in other situations where the impactor cannot be easily moved, measurements would be made in a manner similar to a shaker test.

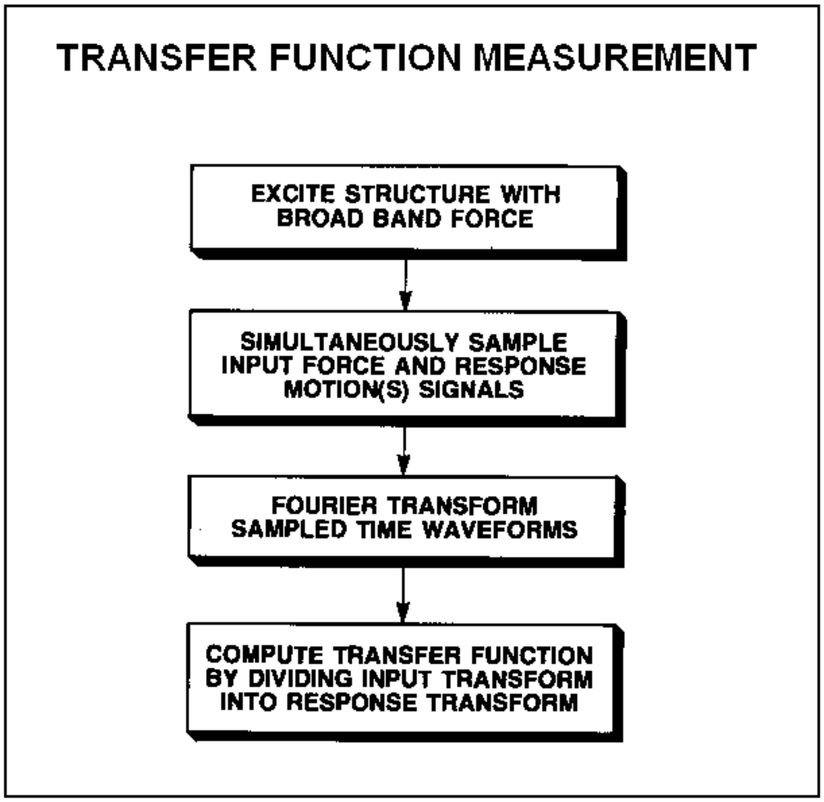
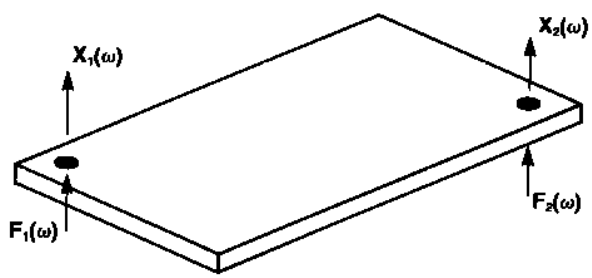


FIGURE 29

MEASURING ELEMENTS OF THE TRANSFER MATRIX

$$\begin{bmatrix} X_1(\omega) \\ X_2(\omega) \end{bmatrix} = \begin{bmatrix} h_{11}(\omega) & h_{12}(\omega) \\ h_{21}(\omega) & h_{22}(\omega) \end{bmatrix} \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}$$



FIRST ROW

$X_1(\omega) = h_{11}(\omega)\, F_1(\omega) + h_{12}(\omega)\, \cancelto{0}{F_2(\omega)}$ $\qquad h_{11}(\omega) = X_1(\omega)/F_1(\omega)$

$X_1(\omega) = h_{11}(\omega)\, \cancelto{0}{F_1(\omega)} + h_{12}(\omega)\, F_2(\omega)$ $\qquad h_{12}(\omega) = X_1(\omega)/F_2(\omega)$

SECOND ROW

$X_2(\omega) = h_{21}(\omega)\, F_1(\omega) + h_{22}(\omega)\, \cancelto{0}{F_2(\omega)}$ $\qquad h_{21}(\omega) = X_2(\omega)/F_1(\omega)$

$X_2(\omega) = h_{21}(\omega)\, \cancelto{0}{F_1(\omega)} + h_{22}(\omega)\, F_2(\omega)$ $\qquad h_{22}(\omega) = X_2(\omega)/F_2(\omega)$

FIGURE 30

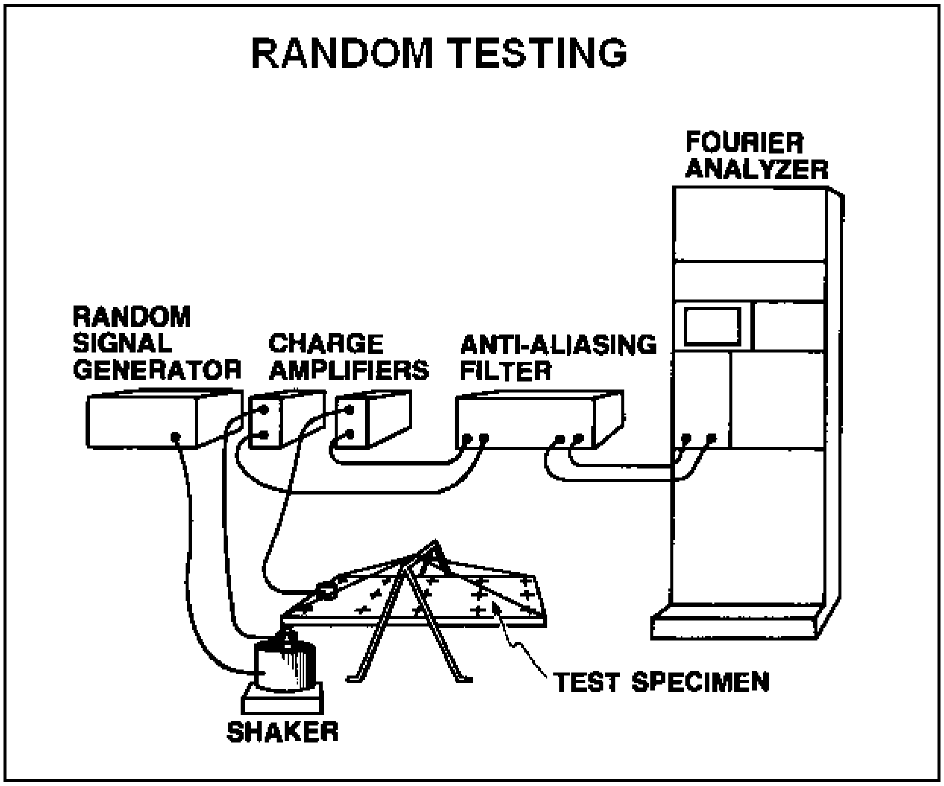


FIGURE 31

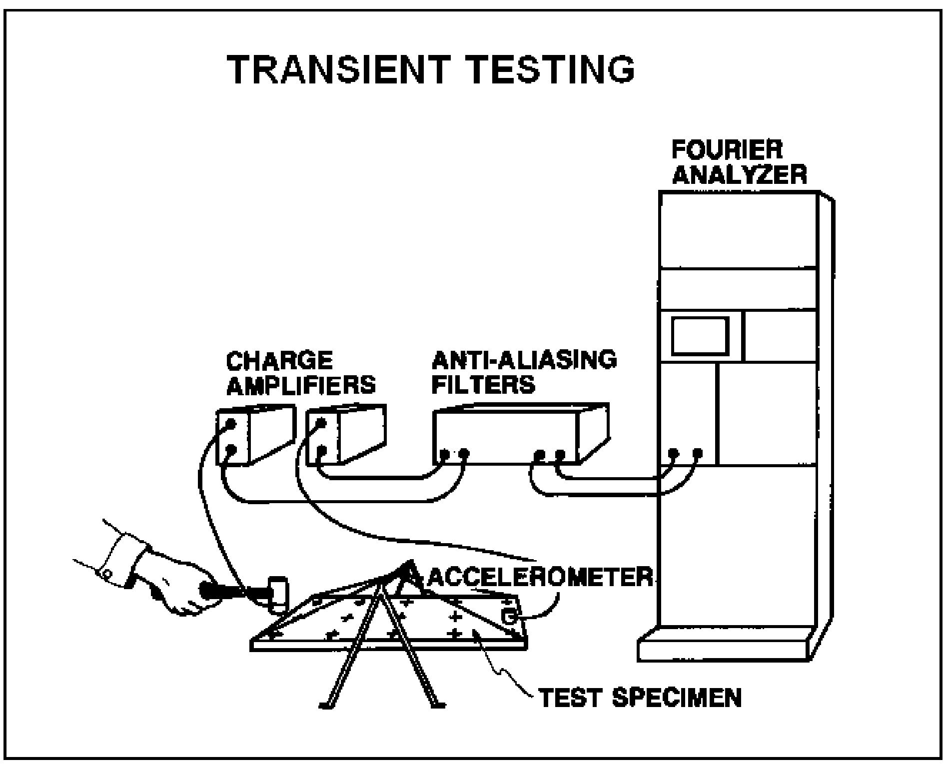


FIGURE 32

Digital Signal Processing Methods

Figure 33 lists some of the signal processing methods that are commonly used to improve the quality of transfer function measurements.

Removing Noise From Measurements

Power spectrum averaging is usually done during transfer function measurement to remove extraneous noise from the measurement. The measurement algorithm depicted in Figure 34 shows how the three power spectrums; input auto power, output auto power, and the cross power spectrum are estimated by way of an averaging scheme. Once enough records of data have been averaged together, the transfer function is computed by dividing the cross power spectrum estimate by the input auto power spectrum estimate. It can be shown that this method yields an unbiased estimate of the transfer function in the presence of noise on the output.

The Coherence function is normally always calculated with the transfer function. As shown in Figure 34 it is computed by dividing the input and response auto power spectrums into the magnitude squared of the cross power spectrum. The coherence function always has values in the range between 0 and 1 and indicates, as a function of frequency, whether the response is being caused by the input (value = 1) or not (value = 0). Values of the coherence less than 1 indicate that an amount of extraneous noise (value = .95 implies a small amount; value = .1 implies a large amount) is being measured with the signals. Low values of coherence usually indicate that more averaging is necessary to remove the noise from the transfer function estimate. Low coherence values also indicated the presence of "leakage" in a measurement. This is discussed later. Figure 35 shows how power spectrum averaging can improve a measurement.

Removing Distortion from Measurements

Distortion, or non-linear motion, is another unwanted contaminant of vibration measurements. Since the dynamic model upon which modal testing is based is linear, we must ensure that our measurements accurately describe linear motion of the structure, and do not reflect any non-linear motion.

Figures 36, 37, 38, 39 and 40 show the results of measuring the transfer function of a single D.O.F. vibrator when its response is both linear, and non-linear or distorted. The distortion was imposed by clipping the response signal with the input amplifier to the ADC. This condition simulates the vibrator being constrained by some kind of "hard stop" which limits its normal response to the input.

Each measurement has the transfer function magnitude of an ideal single D.O.F. vibrator superimposed on it so the measurement can be compared with the expected linear result. The coherence function is also plotted above each measurement.

Figure 36 shows the results of exciting the vibrator with an impulse force. As shown, the linear response of the vibrator is indistinguishable from the ideal linear response. However,

the distorted response varies considerable from the ideal response.

Figure 37 shows the results of exciting the vibrator using a swept sine signal. As shown, the sine wave excitation can cause a considerable amount of distortion which is reflected in a poor quality measurement.

Figure 38 shows the results of using a pseudo random excitation signal. This type of random signal is repeated over every period of measurement so that it is periodic in the measurement window. However it is ineffective for handling distortion, and, as indicated, gives a very noisy measurement when the vibrator undergoes non-linear motion.

DIGITAL SIGNAL PROCESSING METHODS

- ***POWER SPECTRUM AVERAGING*** **REDUCES EXTRANEOUS NOISE**
- ***COHERENCE FUNCTION*** **INSURES VALIDITY OF MEASUREMENT**
- ***RANDOM EXCITATION*** **REDUCES DISTORTION**
- ***PERIODIC SIGNALS*** **AND** ***TIME RECORD AVERAGING*** **REDUCE "LEAKAGE" (SMEARING OF SPECTRUM)**
- ***DIGITAL FILTERING*** **(ZOOM TRANSFORM) IMPROVES FREQUENCY RESOLUTION**

FIGURE 33

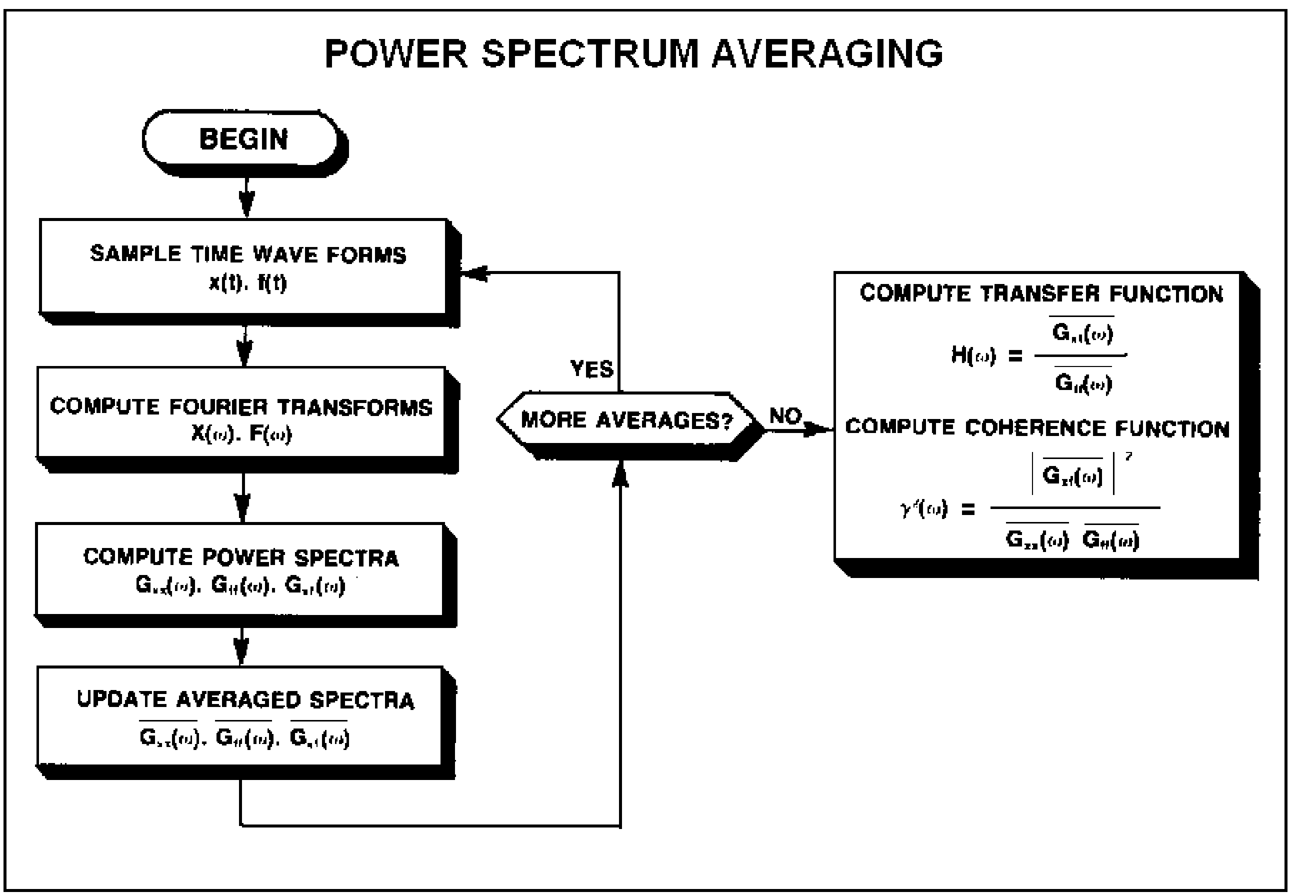


FIGURE 34

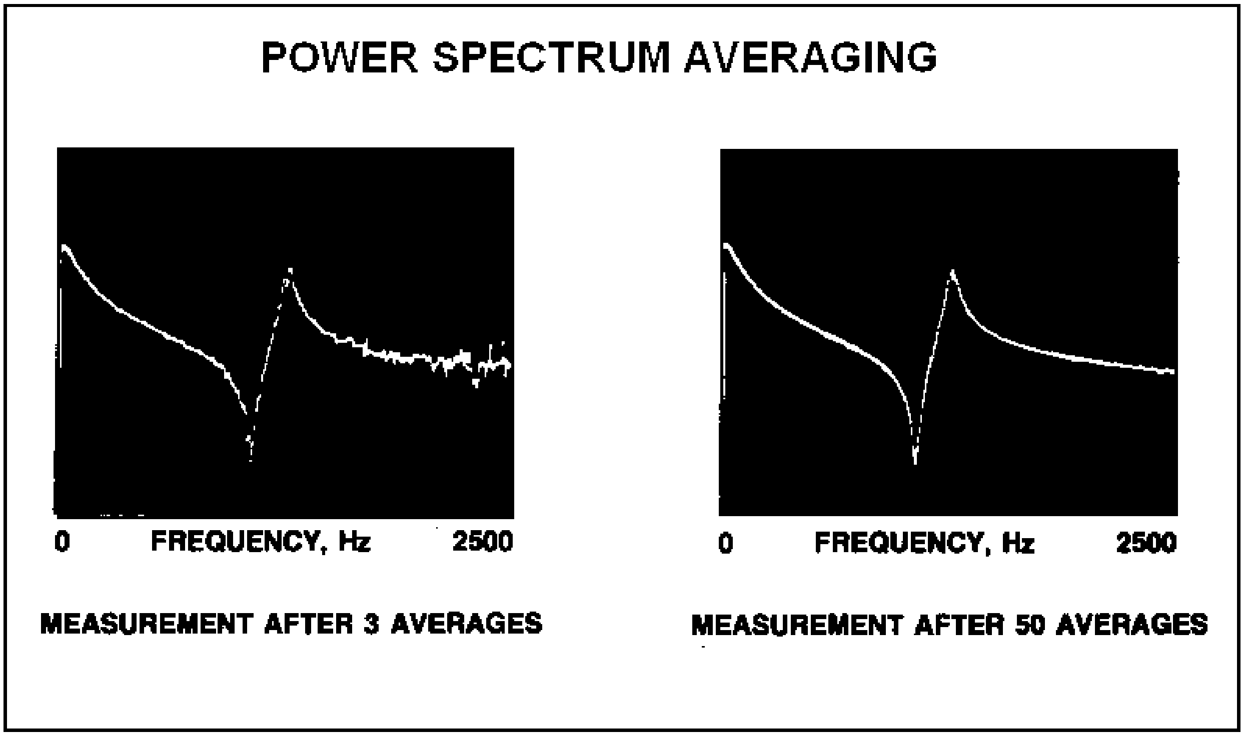


FIGURE 35

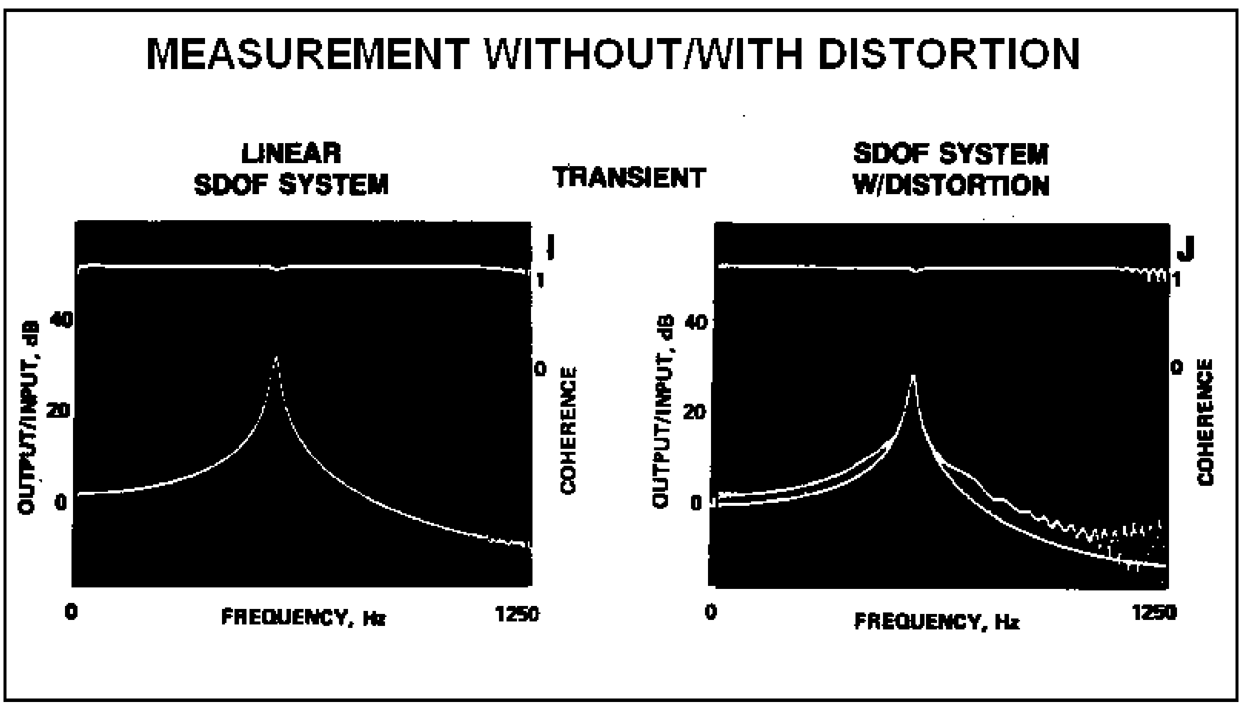


FIGURE 36

Figure 39 shows the measurement results when a pure random signal is used to excite the vibrator. A pure random signal never repeats in the measurement window and therefore excites the vibrator differently during each measurement period. If enough different power spectrum measurements are averaged together, using the previously discussed power spectrum averaging scheme, then the distortion, which shows up as random noise in each power spectrum, can be "averaged out" of the measurement just like any other type of extraneous noise.

When a pseudo random excitation is used, this distortion noise cannot be averaged out since all the power spectrums are exactly alike, i.e. they contain the same distortion components at the same frequencies.

Although a pure random signal is good for removing distortion from a measurement it is not a periodic signal in the measurement the window. Any signal which is not periodic in the measurement window causes a smearing of data in the vicinity of peaks in the spectrum. This error always results when a signal that has been truncated by the measurement window, is Fourier transformed. This smearing of the data is called "leakage". Leakage can be considered as another form of distortion on the data. Since the resulting transfer function waveform has been smeared during the measurement process, resonance peaks will become broader and less peaked than they should be.

Although it appears from Figure 39 that the ideal linear waveform matches the measurement data well, the resulting modal parameters obtained by curve fitting data with leakage in it can be in great error.

Figure 40 shows the results of exciting the vibrator with another type of random signal, called a periodic random signal. This signal contains the desirable properties of both the pseudo random and pure random signals, so that it removes distortion from the measurement without causing leakage.

The periodic random signal is generated as a pseudo random signal so that it is periodic in the measurement window. However instead of continually repeating the same random sequence during each successive measurement period, the pseudo random signal is changed to a different sequence. With this scheme each power spectrum measurement used in the averaging process is the result of a different random excitation just as with pure random excitation. Therefore, distortion is removed by averaging many of these spectra together.

The only drawback of this final test method is that it is 2 to 3 times slower than either of the other random methods, because several time windows of data must be ignored between averages in order to allow the signals from the structure to become periodic in the measurement window.

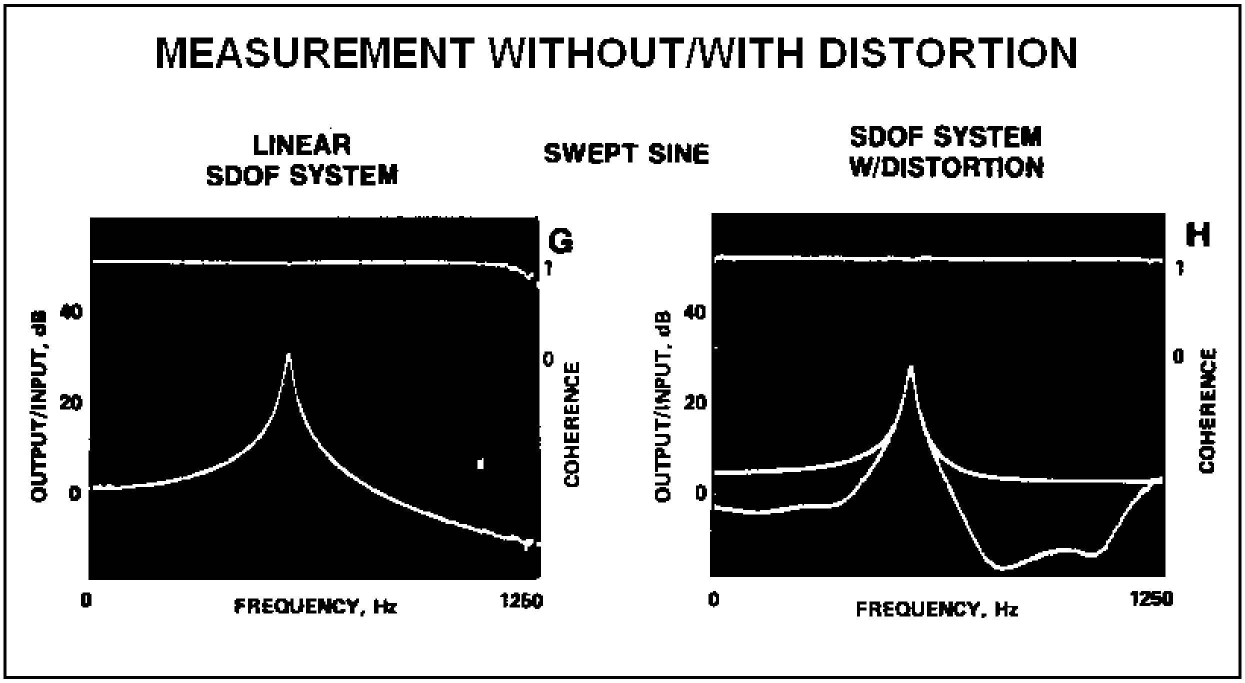


FIGURE 37

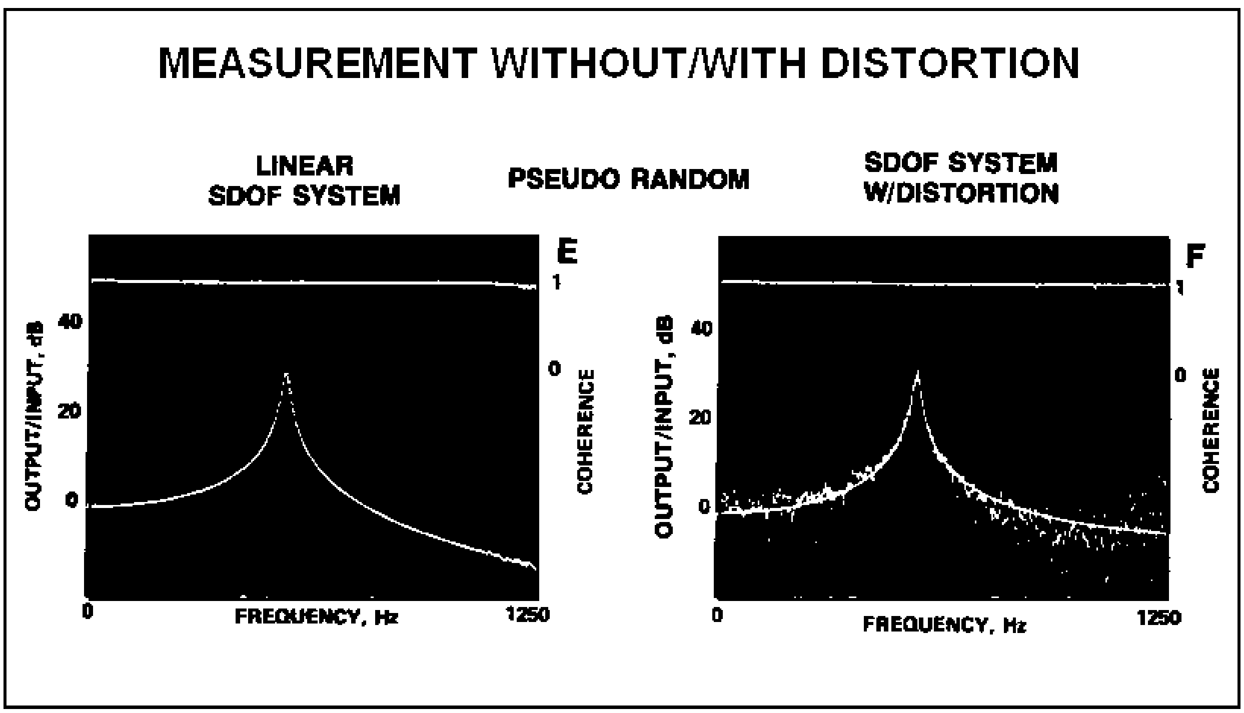


FIGURE 38

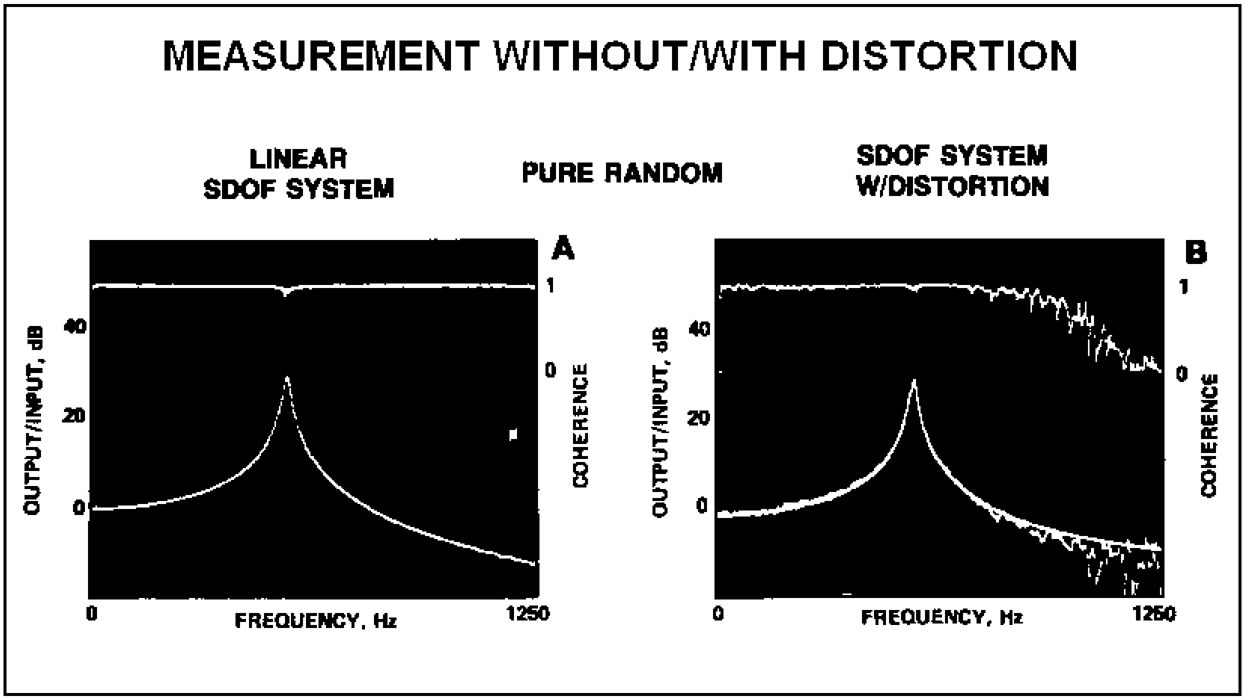


FIGURE 39

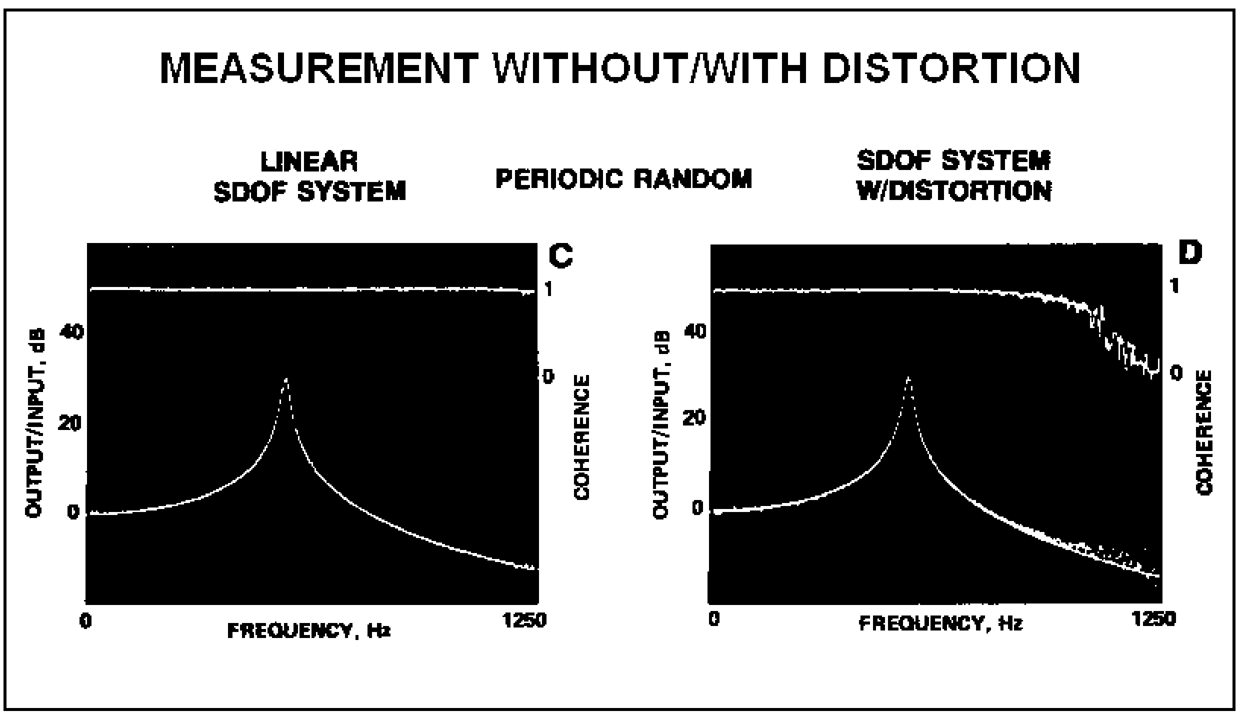


FIGURE 40

Increasing Measurement Resolution

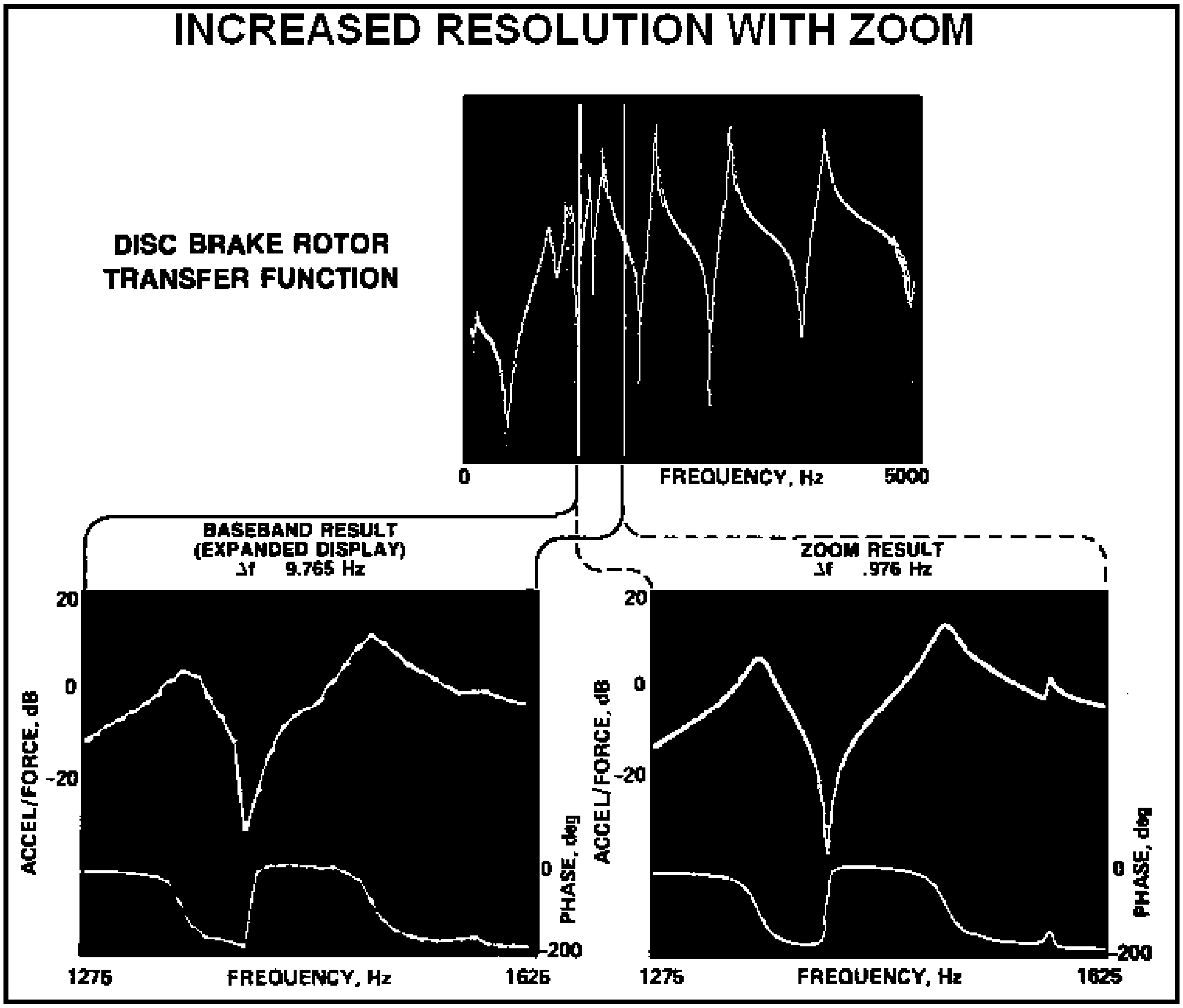
Another measurement capability that is many times necessary in modal testing is to be able to obtain more frequency resolution in the vicinity of modal peaks.

Figure 41 shows a typical "baseband" measurement over the frequency range (0 - 5 kHz). The resolution of this measurement (i.e. frequency difference between spectral lines) is about 10 Hz. The only way to obtain better resolution over this bandwidth is to use a larger memory, collect more data, and compute the spectral functions using more points.

The other method for obtaining increased resolution without using more computer memory is to pass the measurement data through a bandpass filter, and then compute its spectral functions over a smaller frequency band with the same number of frequency lines as in the baseband case. This process of digitally filtering the data and then transforming it to the frequency domain is known as the Zoom transform.

Figure 41 shows a comparison of an expanded view of the baseband measurement in the vicinity of three modes, and a Zoom measurement over the same frequency band. The Zoom measurement has ten times the resolution of the baseband measurement and as a result gives much better definition to the three modal peaks.

Figures 42 and 43 show the improved quality of the Zoom measurement over the baseband measurement in both the CO-QUAD and NYQUIST forms. Figure 44 points out another advantage of the zoom transform; that increased resolution reduces the effects of leakage in a measurement. Large dips in the coherence function in the vicinity of modal peaks are an indication of leakage. As shown, the amount of leakage in the zoom measurement is negligible compared to the baseband measurement.



INCREASED RESOLUTION WITH ZOOM
DISC BRAKE ROTOR TRANSFER FUNCTION
0
FREQUENCY, Hz
5000
BASEBAND RESULT (EXPANDED DISPLAY)
Δf 9.765 Hz
ZOOM RESULT
Δf .976 Hz
20
0
-20
ACCEL/FORCE, dB
0
-200
PHASE, deg
1275
FREQUENCY, Hz
1625

FIGURE 41

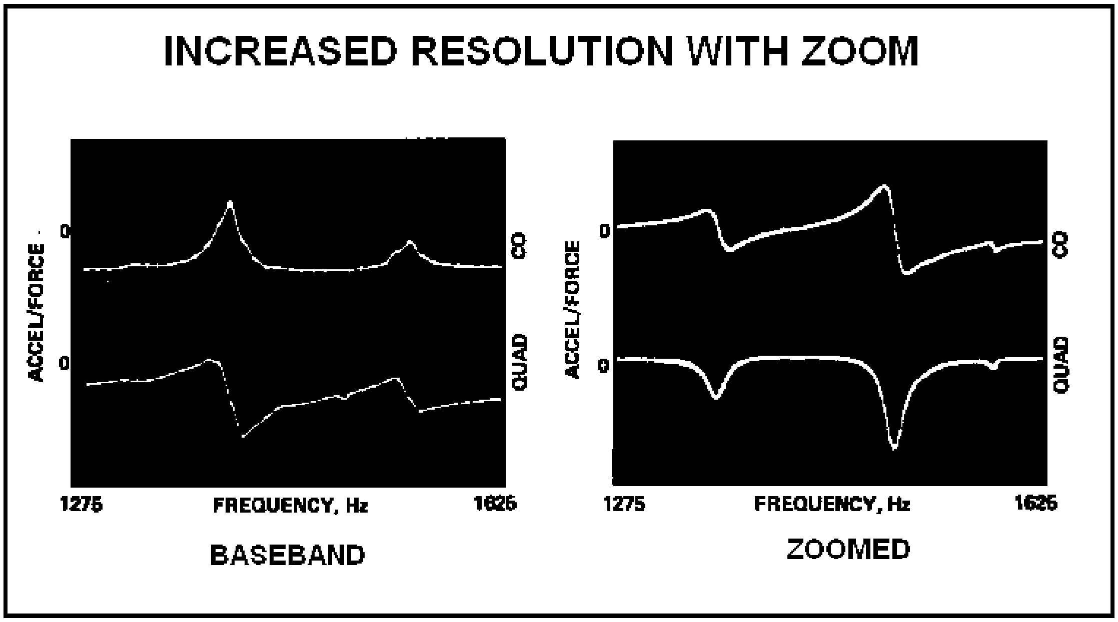

INCREASED RESOLUTION WITH ZOOM
ACCEL/FORCE
0
0
CO
QUAD
1275
FREQUENCY, Hz
1625
BASEBAND
ACCEL/FORCE
0
0
CO
QUAD
1275
FREQUENCY, Hz
1625
ZOOMED

FIGURE 42

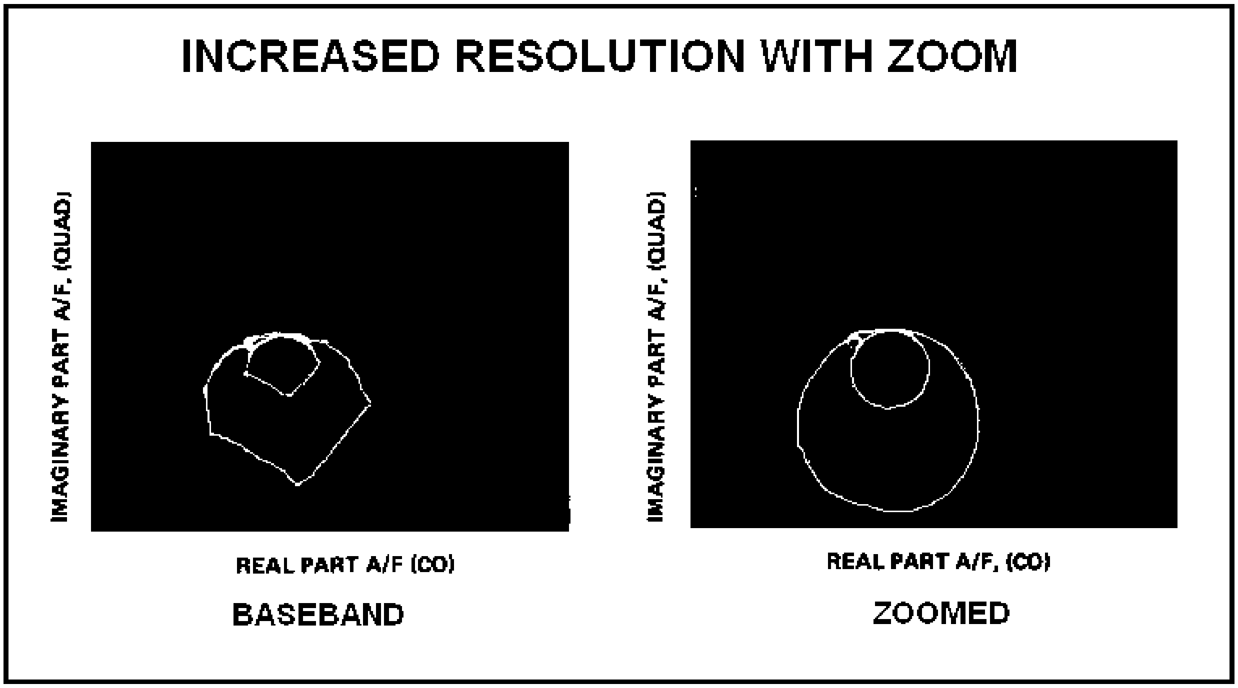


FIGURE 43

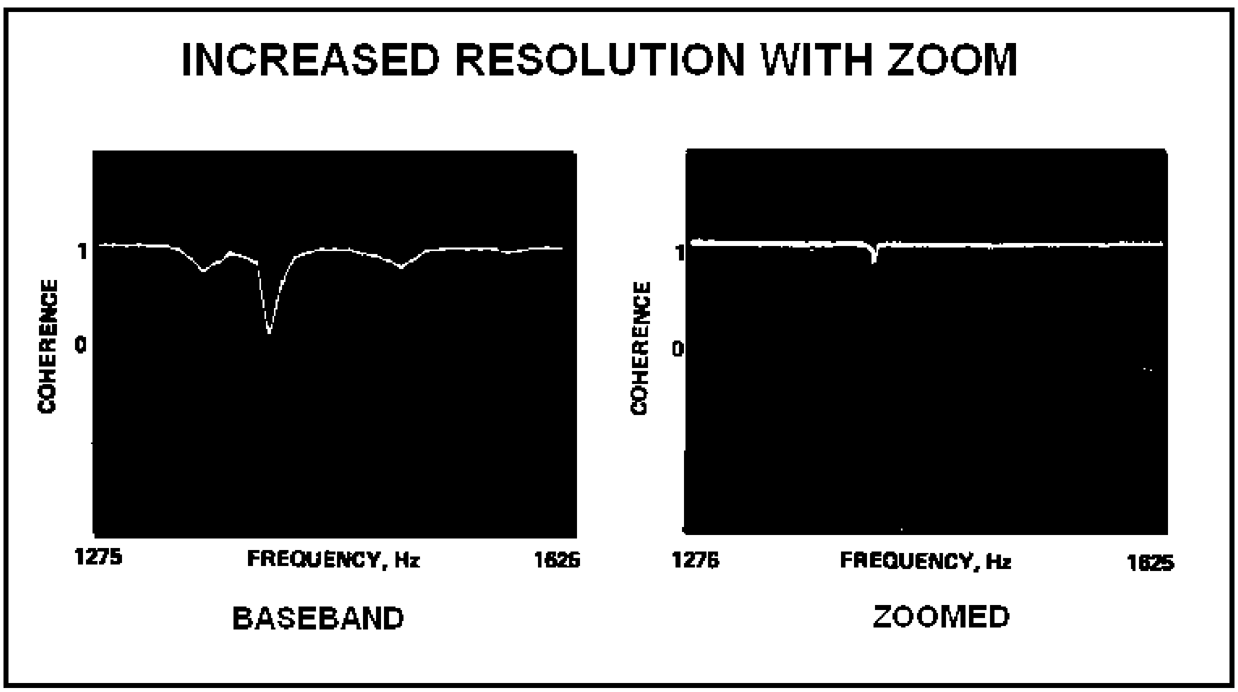


FIGURE 44

MODAL PARAMETER IDENTIFICATION

Once a set of transfer functions has been measured from a structure, modal parameters are identified by "curve fitting" an ideal form for the transfer function to the measurement data.

As shown in Figure 45, at least one row or column of transfer functions from the transfer matrix must be measured in order to identify the mode shapes. The mode shapes, or mode vectors, are then assembled from the identified residues from each measurement at the same modal frequency. This process is depicted in Figure 14.

Figure 46 shows a breakdown of a measurement into a summation of the contributions due to each of the modes of vibration. That is, the magnitude of the transfer function shown by the solid line in the figure, is really the summation of a number of magnitude functions shown by the dashed lines in the figure, each one due to a different mode of vibration.

The modal parameters (frequency, damping and residue), of a single mode can be identified by curve fitting the dashed line function corresponding to that mode. However, since only the solid line function was measured, it is clear that to identify modal parameters accurately, all the parameters of all the modes must be identified simultaneously by some method which fits a multiple mode form of the transfer function to the data. This method is called a "multiple mode" curve fitting method.

Many times, the accuracy of a multiple mode method is not required, so simple, easier-to-use methods known as "single mode" methods are used to identify the unknown parameters. A single mode method treats the data in the vicinity of a modal resonance peak as if it is due solely to a single mode of vibration. In other words the "tails" of the resonance curves from other modes are considered to have negligible contribution to the data in the vicinity of the resonance peak in question.

The amount of error incurred with the use of single mode methods is dictated by the amount of "modal coupling" in the measurements. Figure 48 shows cases of light and heavy modal coupling.

In a light modal coupling case the measurement data in the vicinity of a modal resonance peak is predominantly due to that mode, and the influence of the other modes is minimal. In this case a single mode curve fitting method can give accurate results.

In a heavy modal coupling case the influence of the tails of other modes is not negligible, so a single mode method will incorrectly identify modal parameters.

Curve Fitting Methods

Four different approaches to curve fitting measurements can be taken, as shown in Figure 47. The first two (single mode, and multiple mode) are applied to a single measurement at a time. The second two methods can be used on measurements from multiple rows and columns of the transfer matrix, to obtain better estimates of the modal parameters.

Single Mode Methods

Because of their speed and ease of use, single mode methods should be used whenever sufficiently accurate results can be obtained.

The simplest single mode methods are shown in Figures 49 and 50. Figure 49 shows that modal frequency is simply taken as the frequency of the peak of the transfer function magnitude. Damping can be obtained by measuring the width of the modal peak at 70.7% of its peak value or by computing the slope of the phase function at resonance. The first method is known as the "half power point" method since 70.7% of the magnitude is the same as 50%, or half, of the magnitude squared.

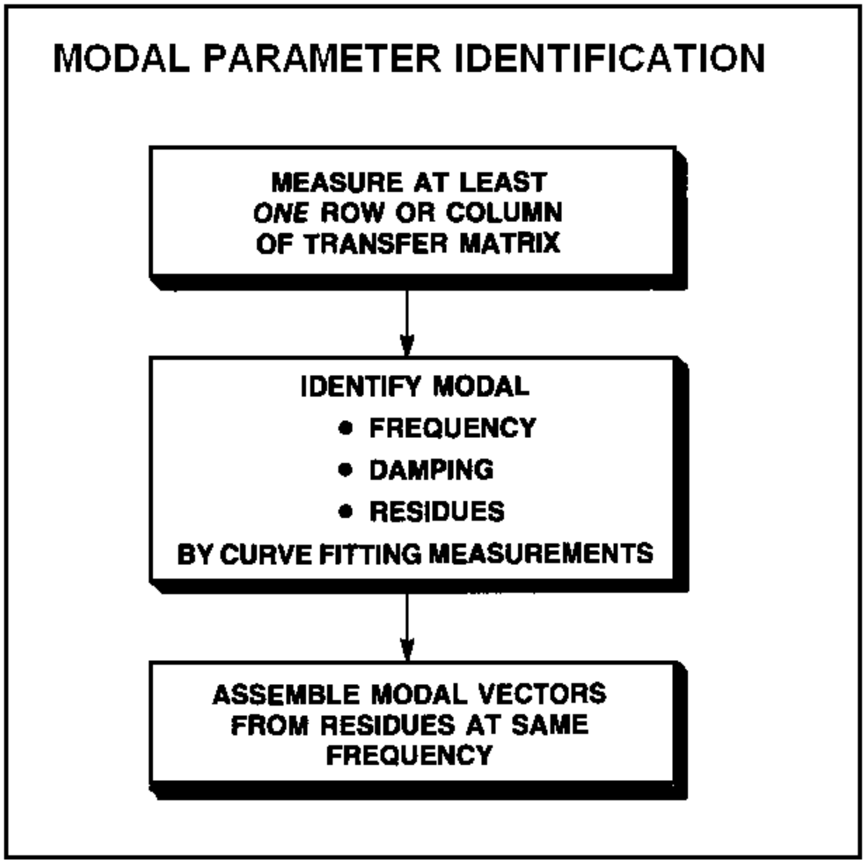


FIGURE 45

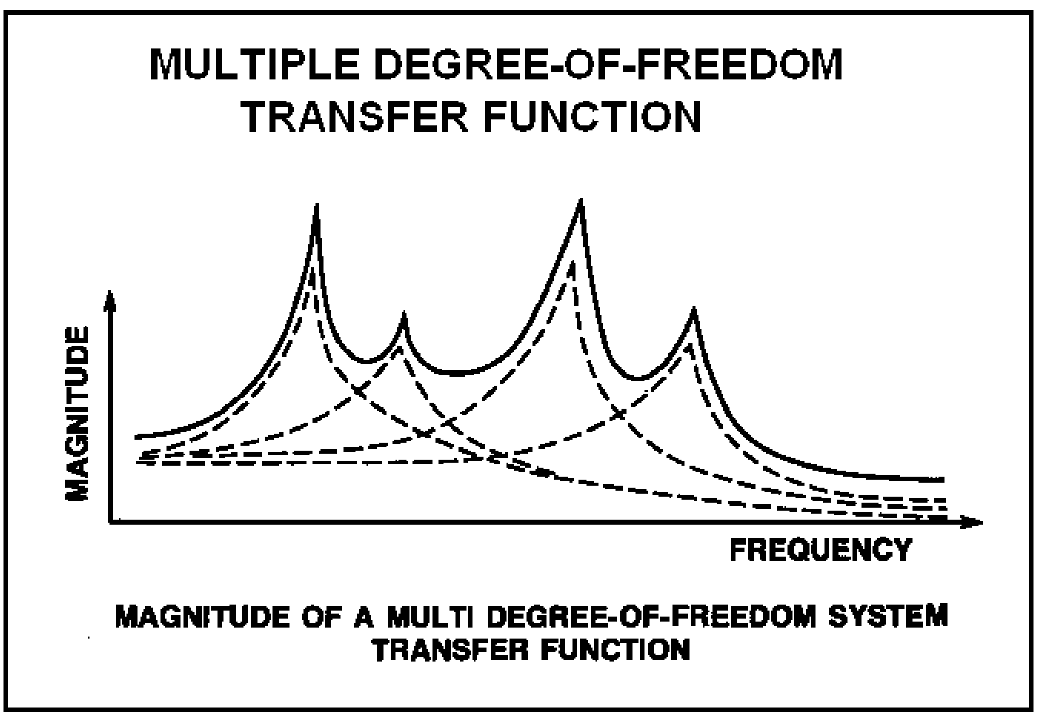


FIGURE 46

CURVE FITTING METHODS

1. SINGLE MODE METHODS
   - FAST
   - EASY TO USE
   - LARGE ERRORS CAN OCCUR WITH HEAVY NOISE OR MODAL COUPLING
2. MULTIPLE MODE METHODS
   - HANDLE NOISE AND MODAL COUPLING WELL IF FREQUENCY RESOLUTION IS SUFFICIENT
   - REQUIRE MORE OPERATOR SKILL
   - CAN GIVE ERRONEOUS RESULTS
3. MULTIPLE ROW/COLUMN RESIDUE SORTING
   - ALLOWS MORE MEASUREMENT FLEXIBILITY
   - IMPROVED ACCURACY OVER SINGLE ROW/ COLUMN RESULTS
   - REQUIRES MORE MEASUREMENTS
4. MULTIPLE MEASUREMENT CURVE FITTING
   - MORE ACCURATE FREQ. AND DAMP. ESTIMATES
   - LARGE AMOUNT OF DATA PROCESSING

FIGURE 47

Finally the residue can be estimated by using the peak value of the imaginary part of transfer function at resonance. This is known as "quadrature picking" or simply the quadrature method.

If the measurements are noisy, or if the frequency resolution is not good, all of the above methods can yield largely incorrect results since they use only one or two data points from the measurement.

The methods shown in Figures 51 and 52 will yield better results in general, since they use more measurement data. The method shown in Figure 51, the so called "circle fitting" method, is a way of estimating the residue by least squared error fitting the parametric form of a circle to the measurement data in Nyquist form. The method shown in Figure 52 is a simple complex division of one measurement into all the other measurements in the set. The result of each divide is a complex constant in the vicinity of a resonance. Several values over an interval of frequencies around the resonance can then be averaged together to obtain an estimate of the residue. These methods and others are explained in more detail in Reference (3)

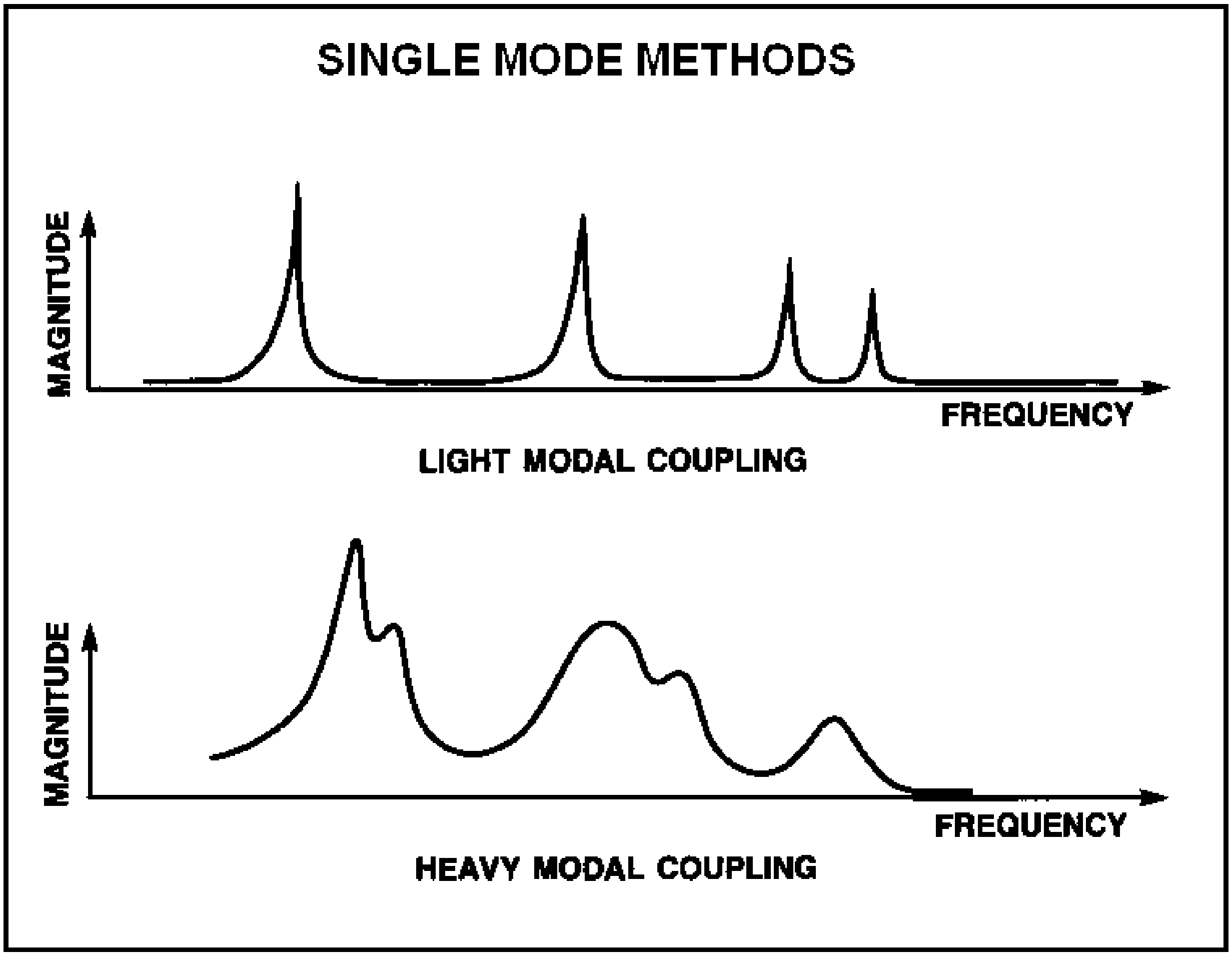

SINGLE MODE METHODS
MAGNITUDE
FREQUENCY
LIGHT MODAL COUPLING
MAGNITUDE
FREQUENCY
HEAVY MODAL COUPLING

FIGURE 48

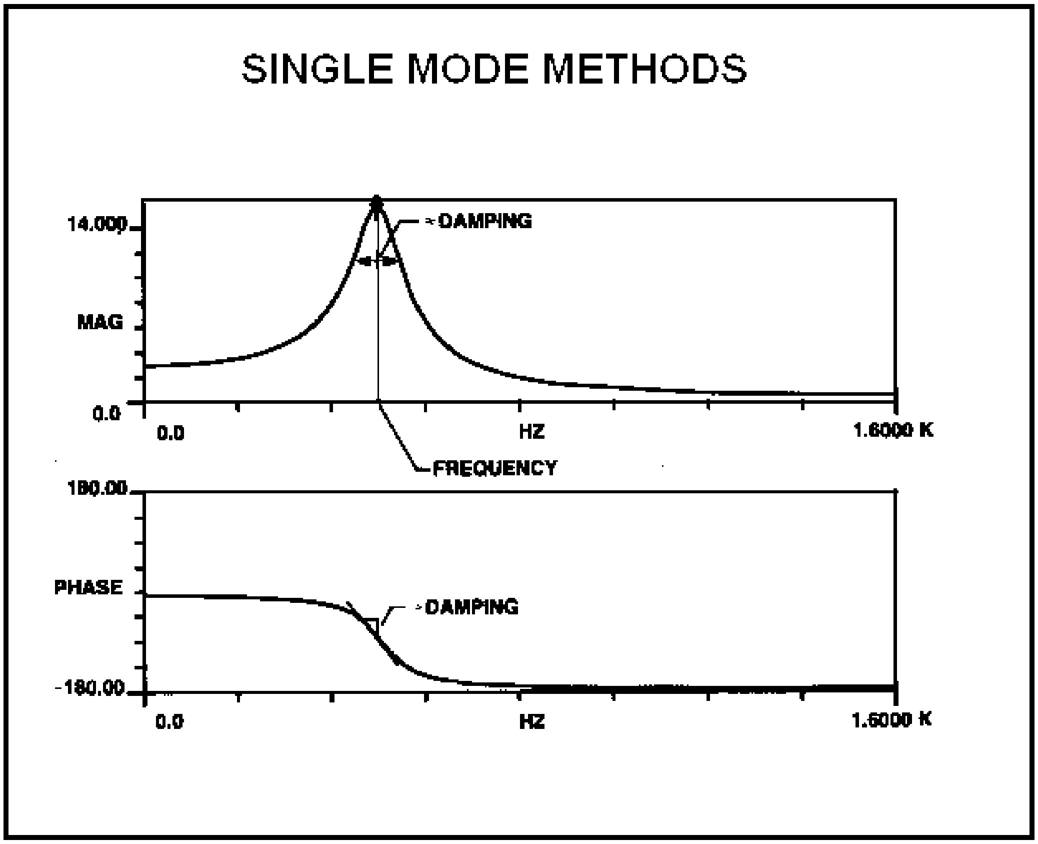


FIGURE 49

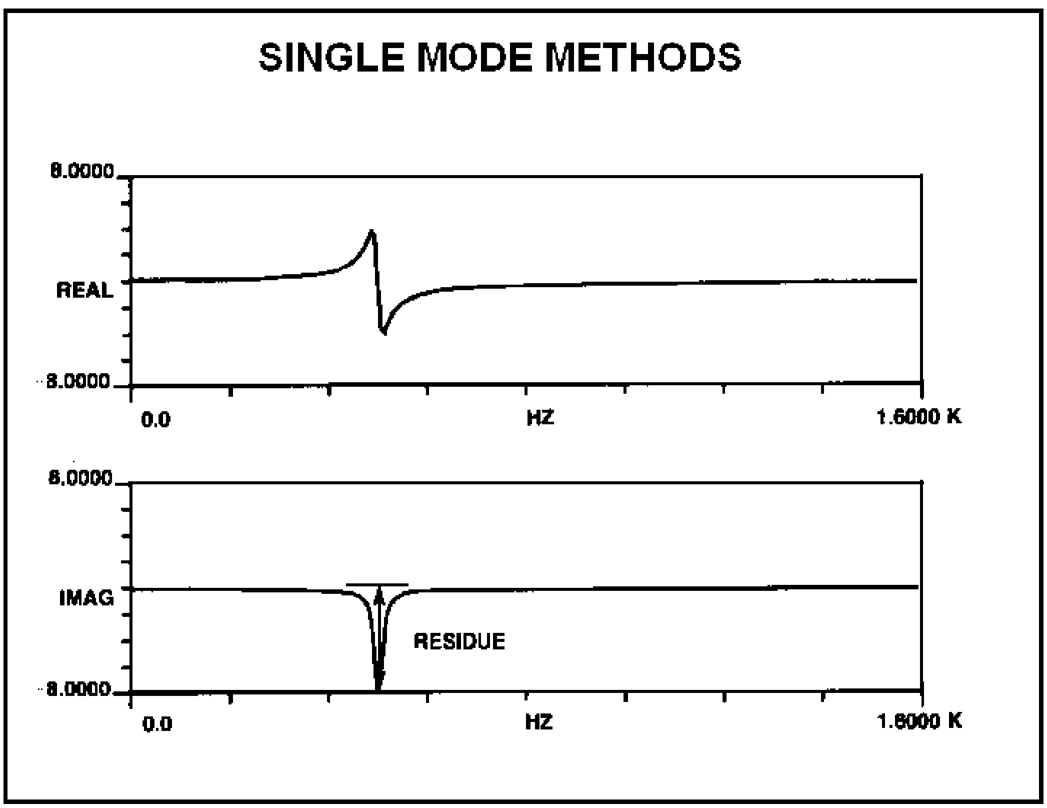


FIGURE 50

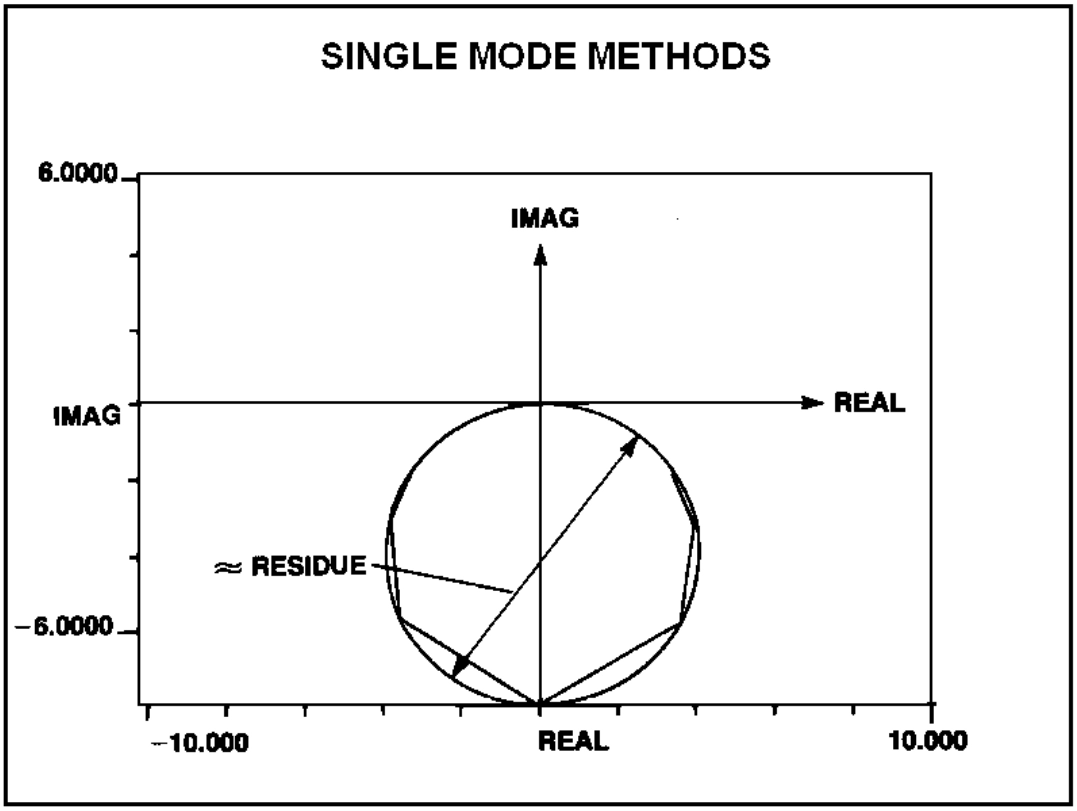


FIGURE 51

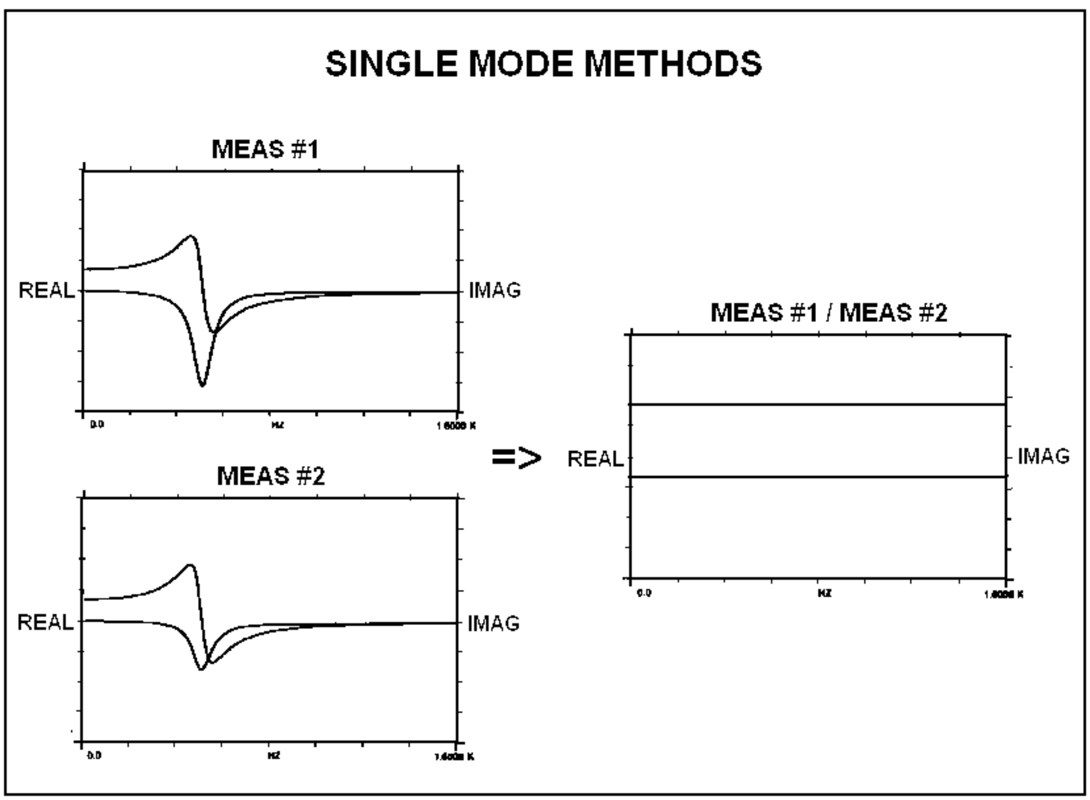


FIGURE 52

Multiple Mode Methods.

Multiple mode methods involve curve fitting a multiple mode form of the transfer function to a frequency interval of measurement data containing several modal resonance peaks. In the process, all the modal parameters (frequency, damping and residue) for each mode are simultaneously identified.

Figure 53 shows two different forms of the transfer function which can be used for curve fitting. If the partial fraction form is used, the residues and poles (frequency and damping) can be identified directly from the curve fitting process. Normally, a least squared error curve fitting procedure is used. This yields a set of linear equations which must be solved for the residues, and a set of non-linear equations which must be solved for the poles. An iterative method is normally used to solve these equations.

If the polynomial form of the transfer function is used, the coefficients of the numerator and denominator polynomials are identified during curve fitting, and a root finding routine must then be employed to find the poles, and partial fraction expansion to find the residues. The advantage of the polynomial form, however, is that the least squared error equations are linear and thus can be solved for the unknown polynomial coefficients by non-iterative methods.

A third method of fitting a parametric form of the impulse response function to impulse data, (obtained by inverse Fourier transforming the transfer function data into the time domain) is also commonly used. This method is called the Complex Exponential method or the Prony algorithm.

The impulse response can be written in terms of modal parameters as

$$\mathbf{x(t)} = \sum_{\mathbf{k=1}}^{\mathbf{m}} |\mathbf{r_k}| \mathbf{e}^{-\sigma_k \mathbf{t}} \,\mathbf{sin\,e}(\omega_{\mathbf{k}} \mathbf{t} + \alpha_{\mathbf{k}})$$

where: m = number of modes

σ_k = damping coefficient

ω_k= natural frequency

$|r_k|$= magnitude of residue

α_k = angle of residue

t = time variable

This form, which can also be written as the sum of complex exponential functions, is curve fit to the time domain data using a very efficient algorithm. (Reference -[13]).

All of these multiple mode methods will yield the same result, as shown in Figure 53; namely a list of modal parameters identified from a particular measurement. The solid line in the transfer function plot of Figure 53 is the ideal fit function superimposed on the data.

MULTIPLE MODE METHODS

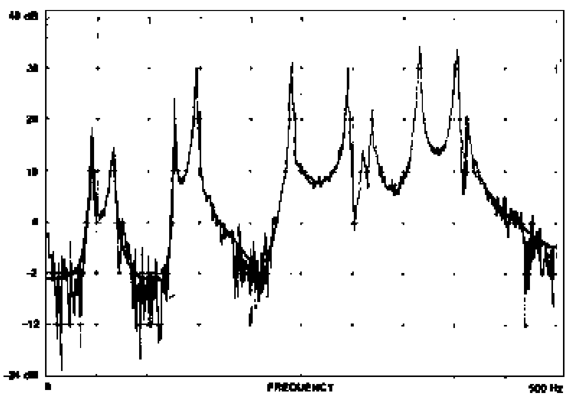
PARTIAL FRACTION FORM

$$H(\omega) = \sum_{k=1}^{n} \left[\frac{r_k}{s - p_k} + \frac{r_k^*}{s - p_k^*} \right] \Bigg|_{s = j\omega}$$

POLYNOMIAL FORM

$$H(\omega) = \frac{a_0 s^m + a_1 s^{m-1} + \cdot \cdot \cdot + a_m}{b_0 s^{2n} + b_1 s^{2n-1} + \cdot \cdot \cdot + b_{2n}} \Bigg|_{s = j\omega}$$

MEASUREMENT & FIT FUNCTION



MODAL PARAMETER ESTIMATES

MODE	FREQ(Hz)	DAMP(%)	RESIDUE MAG	RESIDUE PHS
1	45.5497	.2475	87.1059	186.1
2	66.9754	1.5872	118.0888	2.5
3	129.1113	.1831	109.5194	163.1
4	148.4386	.1003	440.5068	349.8
5	246.5847	.1553	605.3250	354.4
6	250.5989	.2320	97.7994	159.4
7	303.5508	.1004	475.4050	181.0
8	319.2407	.7852	229.3764	182.8
9	328.2293	.2580	286.7471	169.2
10	375.5362	.0579	890.5685	358.1

FIGURE 53

Residue Sorting

Rather than simply assemble residues from one row or column of the transfer matrix into a mode vector, residue estimates from other rows or columns of the matrix can also be used to improve the estimate of the mode vector. This procedure is known as "residue sorting" and is discussed in detail in Reference [4].

Multiple Row/Column Fitting

Many times modal coupling and/or noise on the measurements may

be so great that it is difficult to determine how many modes there are, or to correctly identify their parameters from any single measurement. In these cases a curve fitting procedure that identifies modal parameters from the entire set of measurements should be used.

The last two "multiple measurement" methods are not commonly used today, but will probably be employed more in the future as curve fitting methods continue to improve.

TYPICAL MODAL TESTING EQUIPMENT

Over the past 5 years a number of modal testing systems have been offered for sale by manufacturers of digital signal processing equipment, such as Hewlett-Packard.

The first systems to be offered were of the type described in Figure 54. They are mini-computer based systems with large data and program storage capabilities. The software capabilities offered with these systems are quite extensive, including single and multiple mode curve fitting algorithms as well as normal mode testing software.

More recently, smaller, cheaper and easier-to-use modal test systems, consisting of a two-channel transfer function analyzer interfaced to a calculator with some special purpose modal software, have appeared on the market. These systems have less measurement, data storage and analysis capability than the minicomputer based systems, but because of their portability and ease of use, are becoming very popular for troubleshooting problems.

TYPICAL MODAL TESTING EQUIPMENT

1. MODAL TESTING SYSTEM
   - MINI-COMPUTER BASED
   - MULTI-SHAKER OR TRANSFER FUNCTION METHOD
   - MULTIPLE AND SINGLE MODE CURVE FITTING
   - 4 ADC CHANNELS AND UP
   - LARGE DISC OR TAPE DATA STORAGE
   - ANIMATED DISPLAY OF MODE SHAPES

2. TWO-CHANNEL TRANSFER FUNCTION ANALYZER AND CALCULATOR
   - TRANSFER FUNCTION METHOD
   - SINGLE MODE IDENTIFICATION ALGORITHMS
   - MODERATE DISC OR TAPE DATA STORAGE
   - ANIMATED DISPLAY OF MODE SHAPES

FIGURE 54

REVIEW

This paper has discussed the subject of measuring and modeling the dynamics of mechanical structures. It was shown that the modes of vibration are not only physically observable, and thus are intuitively understandable to most people, but they are also defined by certain fundamental parameters of a linear dynamic model for an elastic structure.

Modes of vibration were also shown to be the link between a mathematical model developed with finite element modeling methods, and an equivalent transfer function model, the elements of which can be measured in a testing laboratory.

Two different methods of performing a modal test were discussed, but the emphasis in this paper was placed on the more recently developed transfer function method, which has been applied to a much wider variety of problems than the normal mode testing method.

In addition to the speed, low cost and versatility of the transfer function method, the capability of displaying mode shapes in animation using this equipment has proved to be a valuable aid for quickly evaluating test results, and in many cases for determining solutions to noise and vibration problems.

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