Identification of the Modal Properties of an Elastic Structure From Measured Transfer Function Data

Mark Richardson and Ron Potter
Digital Signal Analysis
Hewlett-Packard Company
Santa Clara, California

ABSTRACT

A technique for identifying the modal properties of an elastic structure in a testing laboratory is presented. The technique is based upon the use of digital processing and the fast Fourier transform (FFT) to obtain transfer function data, and then the use of a least squared error estimator to identify modal properties from the transfer function data. Both analytical and experimental results are presented.

INTRODUCTION

In recent years the implementation of the fast Fourier transform (FFT) in low cost mini-computer systems has provided the environmental testing laboratory with a faster and more powerful tool for acquisition and analysis of vibration data from mechanical structures. The results are used by analysts and designers alike as an aid to better understanding and improving mechanical designs.

In this paper an analytical technique is presented which as been implemented in a Fourier Analyzer to provide modal data on site in a testing laboratory. The technique is based upon the application of a least squares estimator to measured transfer function data. During the process the natural frequencies, damping factors, and mode shapes of all the predominant modes of vibration of a structure are identified.

A brief review of the modal theory and a derivation of the analytical form used in the estimation process are given in the following section. Following that is a discussion of how parameters are obtained from a single transfer function, and some experimental results are given. Lastly the global nature of a mode is discussed and verified with experimental results.

MODAL THEORY

Assume that an elastic system has n-degrees of freedom and that its motion can be adequately described by n-linear differential equations with constant coefficients, written as

$$M \frac{d^2 x(t)}{dt^2} + C \frac{dx(t)}{dt} + Kx(t) = f(t) \qquad (1)$$

where $x(t)$ and $f(t)$ are displacement and force n-vectors respectively, and M, C, and K are real symmetric matrices. M is called the mass matrix, C the damping matrix, and K the stiffness matrix.

Taking the Laplace transform of equation (1) gives

$$B(s) X(s) = F(s), \quad B(s) = Ms^2 + Cs + K \qquad (2)$$

where $X(s) \leftrightarrow x(t)$ and $F(s) \leftrightarrow f(t)$ are vector Laplace transform pairs. B is defined as the (n x n) system matrix. Eq. (2) is often referred to as an expression of the dynamic flexibility of the structure.

Eigenvectors (y_k) and eigenvalues (λ_k) of the matrix can be defined in the usual way, i.e. to satisfy the equation

$$B y_k = \lambda_k \, y_k \qquad\qquad l \le k \le n \qquad (3)$$

where y_k is an n-vector and λ_k is a constant. The system matrix B has (n) eigenvalues and (n) eigenvectors; each eigenvalue-vector pair is defined by equation (3).

It is straightforward to show that the y_k eigenvectors are orthogonal, provided all values λ_k are different, as follows:

For two different eigenvalues (k) and (j)

$$B y_k = \lambda_k \, y_k \qquad (4)$$

and

$$B y_j = \lambda_j \, y_j \qquad (5)$$

Note that equation (5) can be rewritten as

$$\left(B y_j\right)^t = y_j^t \, B_j^t = y_j^t \, B = \lambda_j \, y_j^t \qquad (6)$$

since B is symmetric. (The superscript t denotes the transpose operation.) Pre-multiplying equation (4) by y_j and post multiplying equation (6) by y_k gives

$$y_j^t \ B \ y_k = \lambda_k \ y_j^t \ y_k = \lambda_j \ y_j^t \ y_k \qquad (7)$$

so

$$\left(\lambda_k - \lambda_j\right) y_j^t \ y_k = 0 \qquad (8)$$

Thus for $\lambda_k \neq \lambda_j$, $y_j^t \ y_k = 0$ which defines orthogonality between two eigenvectors.

As usual, the eigenvalues λ_k can be expressed as roots of the determinant

$$\left\| B - \lambda_k \ I \right\| = 0 \qquad (9)$$

where I is an (n x n) identity matrix, and each value λ_k can be found by solving the polynomial equation defined by (9).

We define a transformation matrix Φ as

$$\Phi = \begin{bmatrix} | & | & & | \\ y_1 & y_2 & \cdots & y_n \\ | & | & & | \end{bmatrix} \qquad (10)$$

where the n columns of Φ are the eigenvectors y_k. We also define a diagonal matrix λ of eigenvalues as

$$\lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad (11)$$

Then, the above definition of eigenvalues and eigenvectors can be expressed in matrix form as

$$B \ \Phi = \Phi \ \lambda \qquad (12)$$

By defining the generalized inverse of Φ as

$$\Phi^{-1} = \left(\Phi^t \Phi\right)^{-1} \ \Phi^t \qquad (13)$$

Equation (12) can be rewritten as

$$B = \Phi \ \lambda \ \Phi^{-1}$$

Since the eigenvectors are orthogonal, if follows that $\Phi^t \Phi$ is (n x n) diagonal with elements y_k^2. Thus, $\Phi^t B \Phi = \Phi^t \Phi \lambda$ is diagonal.

If the eigenvectors are normalized to unit magnitude, so that $\Phi^t \Phi = I$, then $\Phi^t B \Phi = \lambda$. In any case, $\Phi^t \Phi \lambda$ is an (n x n) diagonal form of B. The general form $D^t B D$ is called a congruence transformation, and if the columns of D are orthogonal (so that $D^t D$ is diagonal), it is called an orthogonal transformation. Thus, $\Phi^t B \Phi = \Phi^t \Phi \lambda$ represents an orthogonal diagonalization of B. If the eigenvalues of B are unique, then the eigenvectors are also unique except for an arbitrary normalization constant, so this orthogonal diagonalization of B must be unique to the same extent.

The transfer function matrix H of this linear system (1) is defined as

$$H = B^{-1} = \Phi \ \lambda^{-1} \ \Phi^{-1} \qquad (15)$$

assuming that the indicated matrix inverses exist.

We can also write

$$\Phi^t \ H \ \Phi = \Phi^t \ \Phi \ \lambda^{-1}, \ \ (\text{n x n}) \ \text{diagonal} \qquad (16)$$

Thus, $\Phi^t \ \Phi \ \lambda^{-1}$ is the orthogonal diagonalization of H, which is unique except for normalization constants. Note that both B and H are diagonalized by the same orthogonal transformation.

Recall that the elements of B are quadratic functions of **s**. Both the eigenvalues λ_k and the eigenvector components y_k are generally rather complicated (usually irrational) functions of s. This means that the eigenvector components in the time domain are each changing in some complicated way with respect to one another, and that each corresponding eigenfunction (time domain representation of $\Phi^t \ \Phi \ \lambda^{-1}$) is a complicated time waveform. The only real advantage to this formulation is that each eigenvector distribution is orthogonal with respect to all other eigenvectors.

It is preferable to decompose B or H into a set of time invariant vectors (independent of **s**), and put all **s** dependence into some diagonal representation of the system or transfer matrix. Practical experience indicates that this possibility exists, i.e. physical structures exhibit "**standing wave**" vibration patterns at certain frequencies in which a "**global**" vibration mode shape is associated with each "**resonant**" frequency. We are further encouraged by the fact that the solution of the homogeneous wave equation can be expressed as the product of a time function and a space function. Finally the driving function can be decomposed into a linear combination of these homogeneous solutions, and the complete solution obtained in terms of linear combinations of these homogeneous "**eigenfunctions**". It should be apparent that the key to this desired decomposition of B lies in the solution to the homogeneous equation
Bu = 0.

It is now shown how this homogeneous equation can be solved in terms of the previously defined eigenvalues and eigenvectors of B. We begin by recognizing that each element of $H = B^{-1}$ is a rational fraction in **s**, with denominator given by det|B|. Thus, the roots of this denominator, called the poles of H, are the values of $s = s_k$ for which det|B| = 0. These values of s also satisfy the above homogeneous equation **Bu = 0.**

Each element of H can be expanded into a partial fraction expansion about each pole so that H can be written in the following form:

$$H = \sum_{k=1}^{2n} \frac{a_k}{s - s_k} \tag{17}$$

where the a_k's are matrices independent of **s**. Recall the representation of H in terms of eigenvalues and orthogonal eigenvectors.

$$H = \Phi \, \lambda^{-1} \, \Phi^{-1} \tag{18}$$

Each a_k matrix can be found by multiplying H times $s - s_k$, and then setting $s = s_k$, provided all s_k values are different. Thus,

$$a_k = \left[\Phi \left(s - s_k \right) \lambda^{-1} \, \Phi^{-1} \right]_{s=s_k} \tag{19}$$

Recognize that there are 2n poles because each element of B is of quadratic form. Further, the poles generally appear in complex conjugate pairs, because the elements of M, C, and K are real numbers, and hence each quadratic element of B has real coefficients. If poles do not appear in conjugate pairs, then they must be real.

Now, λ^{-1} is a diagonal matrix whose elements are functions of **s**. Furthermore,

$$\det[B] = \det[\lambda] = \prod_{k=1}^{n} \lambda_k \tag{20}$$

because λ is similar to B ($\lambda = \Phi^{-1} B \Phi$), and hence has the same eigenvalues. Thus, any value of $s = s_k$ which satisfies det [B] = 0, will also force one of the λ's, say λ_k, to zero. Rewriting the eigenvector definition

$$B \, y_k = \lambda_k \, y_k \tag{21}$$

it is clear that $B u_k = 0$ implies that $s = s_k$, $\lambda_k = 0$ and

$$u_k = y_k \left(s_k \right) \tag{22}$$

Conjugating this homogeneous equation gives $B^* u_k^* = 0$, for which $s = s_k^*$, and $\lambda_k = 0$ as before.

Also

$$u^* = y_k \left(s_k^* \right) \tag{23}$$

Thus, $\lambda_k = 0$ for either $s = s_k$ or $s = s_k^*$, and the homogeneous solution at $s = s_k$ is the original eigenvector evaluated at $s = s_k$. Also, the solution corresponding to the conjugate poles $s = s_k^*$ is simply $u_k^* = y_k \left(s_k^* \right) = y_k^* \left(s_k \right)$ which is the conjugate of the solution for $s = s_k$.

Now in the evaluation of a_k, we form the diagonal matrix $\left. \left(s - s_k \right) \lambda^{-1} \right|_{s=s_k}$. However, since λ_k contains a factor $s - s_k$, then $\left. \left(s - s_k \right) \lambda_k^{-1} \right|_{s=s_k}$ will have some well defined values, while all other terms of the λ matrix (which do not contain $s - s_k$ because the λ_k values are unique) will go to zero. Hence, there is only one non-zero element in the λ matrix, whose value is $\left. \frac{\left(s - s_k \right)}{\lambda_k} \right|_{s=s_k}$. As a consequence of this, the expression for a_k involves only the kth column of Φ (which is $u_k = y_k \left(s_k \right)$), and the kth row of Φ^{-1} (which must be $u_k^t / u_k^t \, u_k$, since $\Phi^{-1} \Phi = I$). Thus,

$$\begin{aligned} a_k &= \left. \frac{u_k \, u_k^t}{u_k^t \, u_k} \frac{\left(s - s_k \right)}{\lambda_k} \right|_{s=s_k} \\ &= \left. \frac{y_k \left(s_k \right) y_k^t \left(s_k \right)}{y_k^t \left(s_k \right) y_k \left(s_k \right)} \frac{\left(s - s_k \right)}{\lambda_k} \right|_{s=s_k} \end{aligned} \tag{24}$$

Note that $u_k \, u_k^t$ is an (n x n) symmetric (complex) matrix while $u_k^t \, u_k$ is a complex scalar.

Therefore, the a_k matrix is determined by a mode shape vector u_k, which is simply the solution to the homogeneous system equation $B u_k = 0$ for $s = s_k$. Furthermore, each col-

umn of a_k is this same mode shape vector (within a constant multiplier), and each row is the transpose of the vector. From a measurement standpoint this implies that the same mode shape is obtained regardless of which spatial point is excited or monitored. This pervasiveness of the mode shapes throughout the transfer matrix is verified with experimental results later in the paper.

Returning to the partial fraction expansion of H,

$$H = \sum_{k=1}^{2n} u_k \left[\frac{\left. \frac{(s - s_k)}{\lambda_k} \right|_{s=s_k}}{u_k^t u_k (s - s_k)} \right] u_k^t \qquad (25)$$

We can represent this summation of partial fraction terms in matrix form by defining the following matrices:

$$\Theta = \left[\begin{array}{ccc} | & & | \\ \dfrac{u_1}{\sqrt{u_1^t u_1}} & \cdots & \dfrac{u_{2n}}{\sqrt{u_{2n}^t u_{2n}}} \\ | & & | \end{array} \right] \quad \text{n x 2n} \qquad (26)$$

$$\Lambda^{-1} = \left[\begin{array}{ccc} \dfrac{\left. \frac{s - s_1}{\lambda_1} \right|_{s=s_1}}{s - s_1} & & \\ & \ddots & \\ & & \dfrac{\left. \frac{s - s_{2n}}{\lambda_n} \right|_{s=s_{2n}}}{s - s_{2n}} \\ 0 & & \end{array} \right] \qquad (27)$$

so,

$$H = \Theta \, \Lambda^{-1} \, \Theta^t , \quad \text{(n x n) symmetric} \qquad (28)$$

Φ is called the modal transformation matrix, and $\Phi^t \Phi \Lambda^{-1}$ is the transfer matrix in modal coordinates. Note that the columns of Θ are not orthogonal (even though the parent eigenvectors y_k are orthogonal) because each u_k is evaluated at a different value of **s**. However, the elements of Θ are not functions of **s**. All of the **s** dependence is contained in Λ. Each column of Θ represents a normalized mode shape vector for the corresponding pole of H. It should be apparent that this normalization is arbitrary, and could be absorbed into the Λ matrix if desired.

As discussed previously, the poles of H usually occur in conjugate pairs, and for this case the mode shape vectors associated with the negative poles (lower half of s-plane) are simply the conjugates of the vectors associated with the positive poles. Thus, if Θ_1, is defined as that (n x n) part of Θ associated with positive poles, then Θ_1^* will correspond to the negative poles. Similarly, Λ can be broken into two parts, Λ_1 comprising the positive poles, and Λ_2 comprising the negative poles. H can then be represented

$$H = \Theta_1 \, \Lambda_1^{-1} \, \Theta_1^t + \Theta_1^* \, \Lambda_2^{-1} \left(\Theta_1^* \right)^t \qquad (29)$$

or in partitioned form as

$$H = \left[\Theta_1 \; \Theta_1^* \right] \left[\begin{array}{cc} \Lambda_1^{-1} & 0 \\ 0 & \Lambda_2^{-1} \end{array} \right] \left[\begin{array}{c} \Theta_1^t \\ \left(\Theta_1^* \right)^t \end{array} \right] \qquad (30)$$

Each of these sub-matrices is (n x n) and only Λ_1 and Λ_2 are functions of **s**.

Define

$$A_k = \left. \frac{(s - s_k)}{\lambda_k} \right|_{s=s_k} \qquad (31)$$

It is easy to show that

$$A_k^* = \left. \frac{(s - s_k^*)}{\lambda_k} \right|_{s=s_k^*} \qquad (32)$$

so,

$$\Lambda_1^{-1} = \left[\begin{array}{ccc} \dfrac{A_1}{s - s_1} & & 0 \\ & \ddots & \\ 0 & & \dfrac{A_n}{s - s_n} \end{array} \right] , \text{ and}$$

$$\Lambda_2^{-1} = \left[\begin{array}{ccc} \dfrac{A_1^*}{s - s_1}* & & 0 \\ & \ddots & \\ 0 & & \dfrac{A_n^*}{s - s_n}* \end{array} \right] \qquad (33)$$

and H can be written,

$$H = \sum_{k=1}^{n} \left[\frac{u_k u_k^t}{u_k^t u_k} \right] \frac{A_k}{s - s_k} + \left[\frac{u_k^* u_k^{*t}}{u_k^{*t} u_k^*} \right] \frac{A_k^*}{s - s_k^*}$$

$$= \sum_{k=1}^{n} \frac{A_k \left[\frac{u_k u_k^t}{u_k^t u_k} \right] (s - s_k^*) + A_k^* \left[\frac{u_k^* u_k^{*t}}{u_k^{*t} u_k^*} \right] (s - s_k)}{(s - s_k)(s - s_k^*)} \quad (34)$$

We define the modal mass as the coefficient of s^2 in the denominator of each element of H. However, we recognize that this coefficient is arbitrary, depending on the numerator normalization. Notice that A_k has the dimensions of $(s*mass)^{-1}$, so the numerator should be normalized by dividing by something of the form $A_k s_k$. We can use the rather arbitrary normalization factor $-\left(A_k s_k^* + A_k^* s_k \right)$, so

$$m_k = \frac{1}{-\left(A_k s_k^* + A_k^* s_k \right)}, \text{ modal mass}$$

$$c_k = \frac{s_k + s_k^*}{A_k s_k^* + A_k^* s_k}, \text{ modal damping}$$

$$k_k = \frac{1}{\left(\frac{A_k}{s_k} + \frac{A_k^*}{s_k^*} \right)}, \text{ modal stiffness}$$

This implies that $\left| s_k \right| = \dfrac{k_k}{m_k}$, which is the resonant frequency.

Notice that each element of the H matrix has a different zero in the **s**-plane, depending upon the angle of A_k and u_k at each point, but the poles of each element of H are common, and occur at $s = s_k$ and $s = s_k^*$.

For the special case of zero damping ($c_k = 0$), called the **normal mode** case, we find that $s_k = -s_k^*$ is purely imaginary. Thus, the B matrix becomes real symmetric, and it's eigenvalues and eigenvector components become real. This means that $u_k = u_k^*$, and we can show that A_k becomes purely imaginary, so $A_k = -A_k^*$. In this case, the numerator zero in each element of H goes to infinity, and H becomes

$$H = \sum_{k=1}^{n} \left[\frac{u_k u_k^t}{u_k^t u_k} \right] \left(\frac{1}{m_k s^2 + k_k} \right) \quad (35)$$

Thus it has been shown that two transfer function forms of interest in modal analysis, namely the complex eigenvalue-eigenvector case (eq. 34) and the normal mode case (eq. 35) can be obtained from a more general eigenvalue-eigenvector diagonalization of the system or transfer matrix. In the next section the identification of modal parameters from measured transfer function data using eq. 34 is discussed.

IDENTIFICATION OF MODAL PARAMETERS

The technique used to obtain the results presented here involves the curve fitting of analytical expression (34) to a set of measured transfer function data. The curve fitting is performed in a manner which minimizes the squared difference between the complex data and the complex valued analytical function form, i.e. a least squared error estimate of the data is determined

Recall that according to the modal theory, only one row or one column of the transfer matrix need be measured since all other rows and columns contain redundant information. During the process of determining the least squared estimator for the transfer matrix, complex values of s_k and the residues of one column or one row of the transfer matrix H for all predominant modes of vibration are determined.

For example, the qth column of H would have the residues

$$H_{pq} \left. (s - s_k) \right|_{s=s_k} = A_k \left[\frac{u_{kp} u_{kq}}{u_k^t u_k} \right] \quad \text{p=1, ..., n} \quad (36)$$

After measuring these n residues of H, we form the sum of the squares of these numbers giving

$$\sum_{p=1}^{n} \left[A_k \frac{u_{kp} u_{kq}}{u_k^t u_k} \right]^2 = A_k^2 \frac{u_{kq}^2}{u_k^t u_k} \quad (37)$$

Taking the square root, and normalizing the measured residues by this quantity gives $u_{kp} / \sqrt{u_k^t u_k}$, which are the elements of the normalized mode shape vector. The A_k coefficients are readily found from any residue of H_{pq} by dividing by the product of the pth and q^{th} components of the normalized mode shape vector. The modal system parameters (mass, stiffness, damping) are obtained from A_k and s_k, and the mode shape is given by the u_k vectors (generally normalized by $\sqrt{u_k^t u_k}$).

The pole location of mode (k) in the **s**-plane, also called the complex frequency, can be written in terms of the coordinates

$$s_k = \sigma_k + j\omega_k \qquad (38)$$

Where

$$\sigma_k = \frac{c_k}{2m_k} \qquad \omega_k = \sqrt{\frac{k_k}{m_k} - \left(\frac{c_k}{m_k}\right)^2} \qquad (39)$$

σ_k is called the **damping factor** and ω_k the **natural frequency** of mode (k). Other related and commonly used terms are the **damping ratio** and **resonant frequency**

$$\delta_k = \frac{c_k}{2\sqrt{k_k m_k}} = \frac{\sigma_k}{|s_k|}, \quad |s_k| = \sqrt{\frac{k_k}{m_k}} = \sqrt{\sigma_k^2 + \omega_k^2} \quad (40)$$

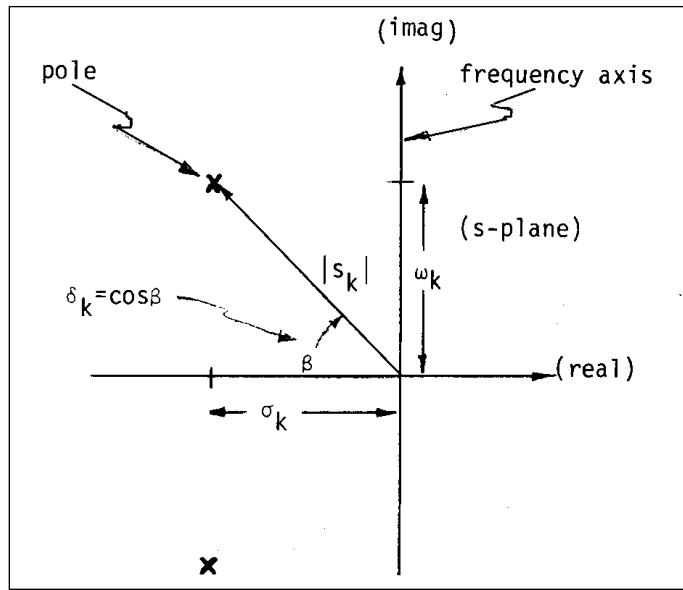These terms are shown in the **s**-plane in Figure 1.

Figure 1. Poles of a Mode

The experimental data was taken from the metal T plate mounted on a foam rubber base shown in Figure 2.
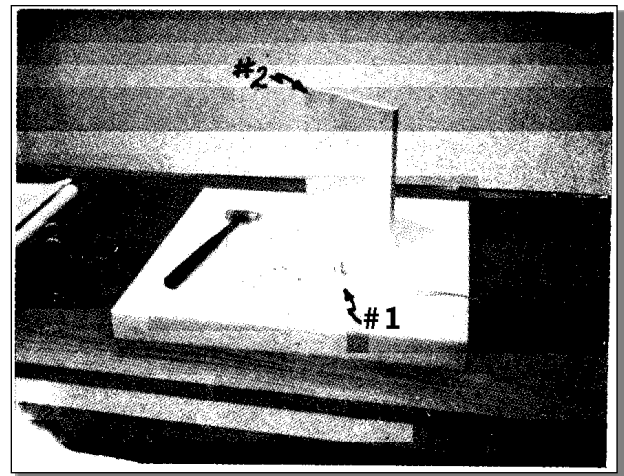

Figure 2. Test Specimen

A hammer was used to provide the broadband excitation force, with a load cell attached to it to measure the force. An accelerometer mounted on the plate was used to measure responses.

The transfer function data was obtained using a Hewlett-Packard 5451B Fourier Analyzer, and the modal parameter identification was performed using the Hewlett Packard Modal Analysis Package.

Transfer functions were measured between 22 different points evenly spaced along the outer periphery of the T-plate. Figure 3 shows a typical transfer function in rectangular or co-quad form.
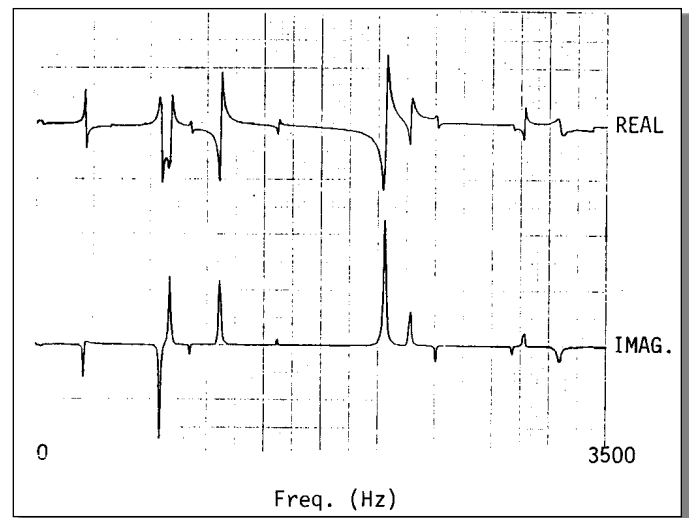

Figure 3. Transfer Function Data

Figure 4 shows the least squares estimate of this transfer function and Table 1 contains its corresponding modal parameters. These results were generated on the Fourier Analyzer using the Modal Package in about 30 seconds.

TABLE 1. MODAL PARAMETERS FROM A SINGLE TRANSFER FUNCTION

Mode	ω (Hz)	$\sigma \dfrac{\text{Rad}}{\text{Sec}}$	Residue	
			Magnitude	Phase Angle (Degrees)
1	305.63	10.63	1.025	166
2	465.50	7.08	0.044	5
3	782.84	29.46	3.135	179
4	842.28	35.95	2.899	358
5	968.96	14.44	0.244	171
6	1156.40	30.84	3.436	0
7	1517.36	11.16	0.246	353
8	2190.07	50.52	7.118	360
9	2349.29	39.04	1.924	3
10	2511.58	22.79	0.405	170
11	2989.90	24.39	0.198	163
12	3061.80	24.77	0.803	359
13	3287.23	81.45	1.372	170

MODE AS A GLOBAL PROPERTY

By far the most fundamental assumption of modal testing is that a mode of vibration can be excited from anywhere on an elastic structure, except of course along its node lines where it can't be excited at all. This is another way of stating the result derived earlier, i.e. that the same mode shape vector (scaled by a different component of itself) is contained in every row and column of the transfer matrix. In addition, modal frequency and damping are constants which can be identified in any element of the transfer matrix, i.e. any transfer function taken from the structure.
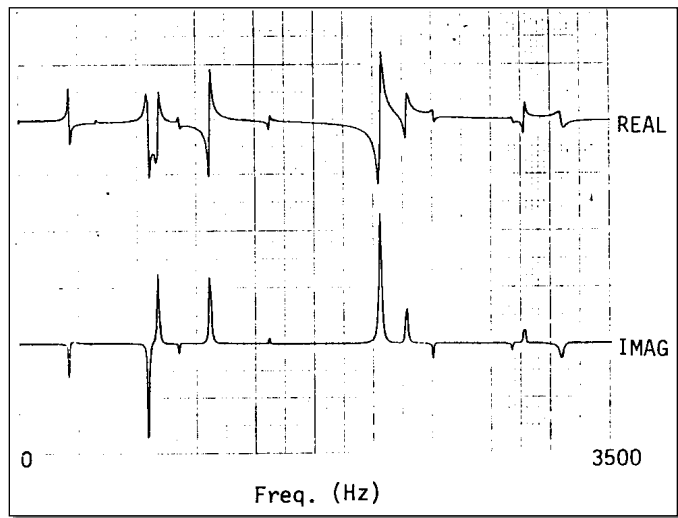

Figure 4. Analytical Transfer Function

It is important to recognize that this global mode shape concept implies some sort of spatial boundaries, beyond which vibrations will not readily propagate. Any attempt to extend B or H beyond these boundaries will result in singular matrices,

and a breakdown of the modal concept. This behavior implies that B and H must be partitioned into sub-matrices some of which will be nonsingular, and will possess well-defined vibration modes. If two linear systems are completely isolated, then a single composite mode including both systems is not meaningful. Conversely, it is important to include enough spatial points to describe all of the vibration modes of interest. If some region of a bounded system is not monitored or excited, or if points are not chosen sufficiently close together, then some modes cannot be adequately represented.

Following are the results of two separate modal tests that were performed on the T-plate. In Test #1 the accelerometer was mounted on the bottom plate as shown in Figure 2 and the plate was impacted with the hammer at the 22 peripheral locations. Using the Fourier Analyzer, a transfer function was measured between each of the 22 impact points and the single response point (accelerometer location). Since the transfer function is the same between two points regardless of which one is the excitation or response point, this test is equivalent to impacting the plate in one spot and moving the accelerometer to all 22 positions. This reciprocity or symmetry assumption is also fundamental to modal analysis and is reflected in the symmetry of the system and transfer matrices.

Test #2 was the same as Test #1 except that the accelerometer was mounted at position #2.

Table 2 contains the least squared estimates of the natural frequency and damping factor of a single mode from the 22 transfer function measurements. These are remarkably good when one considers that the resolution between spectral lines is 10 Hz. Working in a narrower bandwidth or using more data points to describe each transfer function should give better results. Figure 5 is an isometric view of the mode shape in a displaced position.
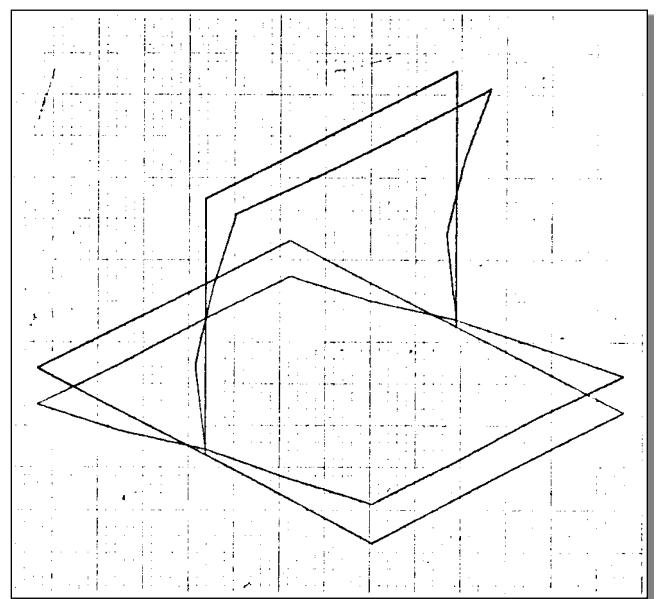

Figure 5. Mode Shape of 305.6 Hz Mode
TABLE 2. MODAL FREQUENCY AND DAMPING

| Meas. | Test # 1 | | Test # 2 | |
No.	Freq (Hz)	Damping (rad/sec)	Freq (Hz)	Damping (rad/sec)
1	306.10	12.09	306.07	11.44
2	305.72	11.12	305.99	10.57
3	305.72	11.22	305.85	10.44
4	305.63	10.62	305.83	10.61
5	305.57	11.09	306.00	9.98
6	305.63	11.06	305.81	10.34
7	305.62	11.05	306.00	10.46
8	305.84	10.81	306.48	11.09
9	305.69	9.75	306.33	11.12
10	305.65	10.00	306.24	10.88
11	305.68	9.93	306.31	10.73
12	305.61	10.11	306.33	10.59
13	305.64	9.84	306.35	10.48
14	305.56	9.85	306.43	10.59
15	305.80	10.54	306.58	9.45
16	305.77	11.83	306.52	11.89
17	305.73	11.03	306.32	11.39
18	305.72	10.89	305.82	10.64
19	305.63	10.71	305.76	9.46
20	305.34	10.02	306.02	9.98
21	305.55	9.76	306.12	8.40
22	305.33	10.09	306.52	10.80

Table 3 contains the corresponding normalized mode shape vectors from the two tests.

CONCLUSIONS

The results indicate that by applying an analytical transfer function expression through least squares estimation to measured data from linearly behaving (small displacements) structures, modal parameters consistent with the theory can be obtained. The vibrations specialist must be continually aware however of the important assumptions necessary for obtaining valid modal results from test specimens.

TABLE 3. NORMALIZED MODE SHAPES

| Position | Test # 1 | | Test # 2 | |
	Mag.	Phase	Mag.	Phase
1	.050	167	.068	167
2	.160	165	.146	167
3	.272	165	.272	166
4	.289	165	.278	166
5	.276	166	.295	164
6	.289	165	.282	163
7	.148	166	.155	165
8	.045	168	.052	170
9	.148	345	.169	348
10	.271	345	.299	347
11	.286	345	.279	347
12	.265	345	.307	348
13	.268	345	.286	350
14	.137	346	.162	350
15	.078	345	.077	344
16	.062	166	.058	171
17	.268	164	.262	166
18	.277	165	.230	166
19	.265	165	.207	164
20	.233	162	.219	164
21	.055	162	.045	157
22	.078	345	.082	353

REFERENCES

1. Foss K.A "Co-Ordinates which Uncouple the Equations of Motion of Damped Linear Dynamic Systems" J. of Applied Mechanics ASME paper 57-A-86, 1958

2. Flannelly, W.G. McGarvey, J.H., Berman, A., "A Theory of Identification of the Parameters in the Equations of Motion of a Structure Through Dynamic Testing" paper No. C-1 Symposium on Structural Dynamics, U. of Technology, Loughborough, England, March 1970

3. Thoren, A.R. "Derivation of Mass and Stiffness Matrices from Dynamic Test Data", AIAA paper 72-346, 13th Structures, Structural Dynamics, and Materials Conf., San Antonio, Texas April 12, 1972

4. Hurty, W.C. and Rubinstein, M.F. "Dynamics of Structures" Printice-Hall Inc., Englewood Cliffs, N.J., 1965

5. Caughey, T.K., O'Kelly, M.E.J. "General Theory of Vibration of Damped Linear Dynamic Systems" Cal. Inst. of Tech., Dynamics Lab., Pasadena, California, June 1963

6. Klosterman, A.L., "On the Experimental Determination and Use of Modal Representations of Dynamic Characteristics", Ph.D. Dissertation, University of Cincinnati, 1971