Modal Testing using Multiple References

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ABSTRACT

With the increased availability of multi-channel FFT analyzers, more emphasis has been placed in recent years on the development of new methods for modal testing using multiple references. In the case of shaker testing, this means the use of two or more shakers. In the case of hammer (impact) testing, this means the use of two or more reference transducers.

This paper presents the results of our recent experiences in both the collection of FRF data, and the curve fitting of it to obtain the modal parameters of a structure. The key difference between these results and those of a more traditional modal test is that multiple rows or columns of the structure's FRF matrix are not only collected simultaneously, but are also curve fit in a simultaneous manner.

In this paper, the advantages offered by multiple reference modal testing and curve fitting are addressed. The theory and usefulness of a frequency domain curve fitting algorithm are extended to account for the effects of "out-of-band" modes, which always occur in practical applications. Verifications of the method and its implementation using both analytical and experimental FRF data are also presented in the paper.

NOMENCLATURE

n = number of DOFs of the model
m = number of modes
r = number of reference points of the FRF measurements
f = number of frequency points used from FRF measurements
t = time variable
s = Laplace variable

$[M]$ = mass matrix $(n \text{ by } n)$
$[C]$ = damping matrix $(n \text{ by } n)$
$[K]$ = stiffness matrix $(n \text{ by } n)$

$\{x(t)\}$ = vector of displacements $(n \text{ by } 1)$
$\{x'(t)\}$ = vector of velocities $(n \text{ by } 1)$
$\{x''(t)\}$ = vector of accelerations $(n \text{ by } 1)$
$\{f(t)\}$ = vector of externally applied forces $(n \text{ by } 1)$

$\{X(s)\}$ = vector of Laplace transforms of displacements $(n \text{ by } 1)$
$\{ICs\}$ = vector of initial condition terms $(n \text{ by } 1)$

$[B(s)]$ = system matrix $(n \text{ by } n)$
$[H(s)]$ = transfer matrix = $[B(s)]^{-1}$ $(n \text{ by } n)$
$[H(t)]$ = r columns of the matrix of impulse responses $(n \text{ by } r)$
$[h(s)]$ = transfer matrix in principle coordinates $(m \text{ by } r)$

$[L]$ = matrix of modal participation factors $(n \text{ by } r)$
$[U]$ = matrix of mode shapes $(n \text{ by } n)$
$[V]$ = orthonormal matrix of principle components $(n \text{ by } m)$
$[v]$ = mode shape matrix in principle coordinates $(m \text{ by } m)$
$[a],[b],[c]$ = unknown constant matrices $(m \text{ by } m)$

ob = subscript denoting out-of-band mode terms
ib = subscript denoting in-band mode terms

p_k = pole location for the k^{th} mode = $-\sigma_k + j\omega_k$
σ_k = damping of the k^{th} mode
ω_k = damped natural frequency of the k^{th} mode, $k = 1, \ldots, m$

INTRODUCTION

A great deal of research and development has been conducted during the past ten years into methods for identifying modal parameters from frequency response function (FRF) measurements which have been taken from a structure. All of these identification methods are commonly referred to as "curve fitting" methods. Single mode (or SDOF) methods were developed first, followed by multiple mode (or MDOF) methods, and then by GLOBAL curve fitting methods [7]. All of these methods operate on a set of FRF measurements which are taken from a **single** reference point on the structure. Another way of saying this is that only data from a single row or column of the matrix of possible FRF measurements is used by these methods.

From an examination of the FRF matrix, written in terms of modal parameters, it is straightforward to show that a single row or column is sufficient for identifying all of the modal parameters, provided that the following assumptions are satisfied:

Assumptions for Single Reference Testing

(1) The structure's motion is linear and symmetric

(2) All of the modes are adequately excited at the reference point

(3) The frequency and damping of each mode are "sufficiently" different from all other modes

Assumption (1) means that the structural motion can be adequately represented by a set of second-order linear differential equations, with symmetric mass, damping, and stiffness matrices. This also implies that the FRF matrix is symmetric. This condition, called Maxwell's Reciprocity, can be easily violated when testing large structures, and is often due to the nonlinear behavior of the structure.

Assumption (2) can be satisfied by choosing a reference point where all of the mode shapes (or at least the mode shapes of interest) are not near their nodal points (or zero points). This can be a difficult assumption to satisfy with a single reference point, and an important mode may be missed.

Assumption (3) has to do with the ability of the curve fitting method to correctly identify the modal parameters of very closely spaced modes. Some structures may in fact have two or more modes with the same frequency and damping values, (i.e. repeated roots). These parameters cannot be correctly found with a single reference curve fitting method. More often, though, in practical situations we encounter measurements which have a combination of poor frequency resolution and modes which are so close in frequency and damping that the curve fitting method cannot resolve them.

With multiple reference modal testing, **all of the above assumptions can be relaxed.** Assumption (1) can be satisfied, especially for large structures, by exciting them with multiple shakers. This, plus the use of random excitation signals, frequency spectrum averaging (a standard capability of all FFT analyzers), and the required inversion of the input power spectrum matrix, will yield a set of symmetric FRF measurements which are a best approximation, in a least squared error sense, of the linear motion of the structure.

Assumption (2) is more easily satisfied when multiple references are used. It is still possible, however, to pick excitation (or response) references so that modes are missed. This can still be a particular problem on structures which have many local mode shapes, (shapes with many node points).

Assumption (3) can be largely overcome in most practical testing situations by using multiple reference testing and curve fitting. Each additional row or column of the FRF matrix allows us to resolve the difference between another pair of repeated modes. Hence, with two references, two repeated modes can be correctly identified; with three references, three repeated modes can be correctly identified.

In summary, then, multiple reference excitation provides a more uniform distribution of energy to non-linear structures, and the use of random signals and frequency domain signal averaging yield more consistent FRF measurements [8].

Additionally, multiple reference measurements reduce the likelihood of "missing" a mode during the curve fitting process. Multiple reference curve fitting offers its biggest advantages when the FRF data contains closely coupled (closely spaced and/or heavily damped) modes, or modes which are truly repeated.

Experimental modal analysis using multiple references has attracted considerable attention lately, as a research subject. In his recent thesis [6], J. Leuridan named this overall approach to parameter identification the "Direct Parameter Identification" method. He has categorized a variety of time-domain and frequency-domain approaches for solving this problem.

In this paper we have extended the usefulness of a frequency-domain curve fitting technique which was first published in 1985 [2], [9]. Zhang, et al [10] modified this technique by adding initial condition terms, which gave more accurate results. Lembregts, et al [5] further improved the method by adding conjugate pole terms to the curve fitting model.

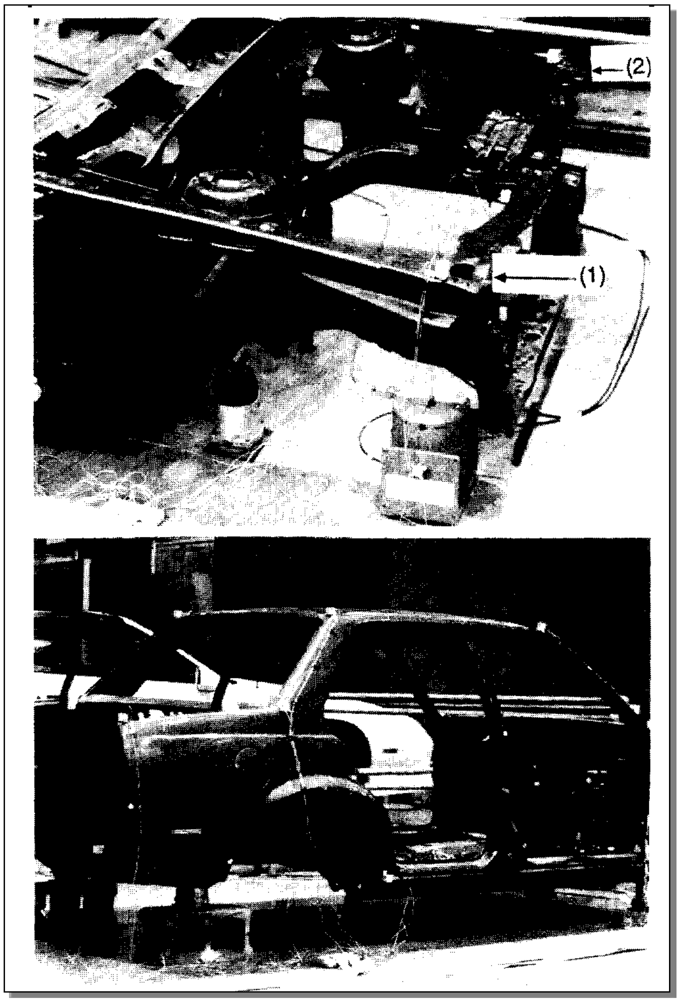
In many practical applications, estimates of the parameters of a relatively small number of modes are needed from data that may contain the influential effects of many other "out-of-band" modes. In this paper, we had extended the method described in [5] to include additional terms which compensate for these out-of-band influences.

We have found that this generalized form of the curve fitting equations consistently yields better modal parameter estimates than the previous forms, especially its damping estimates. Verifications of the method and its implementation using both analytical and experimental FRF data are presented in the paper. Also included is a comparison of the results of this method with some analytical data for a square plate with a repeated mode, and a comparison with results from the previous method [5].

DATA ACQUISITION

The most important goal of any modal test is to obtain accurate modal parameters. Typically these parameters are obtained from experimentally measured FRFs through the process of parameter estimation, or "curve fitting". The success of this process depends upon many factors. However, before any testing is begun, some basic questions should be answered:

Figure 1 Two-Shaker Test of Body-in-White



(1) How well should the FRF measurements describe the dynamic characteristics of the structure under test? What frequency range and resolution are required? Which modes are important?

(2) What is the required accuracy of the results? Will the results be used for comparison with those of a finite element model? Will they be used as the basis of a model which will evaluate methods of modifying the dynamic behavior of the structure? Or will the results be used just to get "an idea" of what's going on?

(3) How well will the available curve fitting methods extract the dynamic characteristics (modal parameters) from the FRF measurements?

In the majority of cases today, the data acquisition phase of an experimental modal analysis is performed with only one reference location. This equates to measuring one row or column of the experimental FRF matrix. The reason for this may be because of a limited amount of test hardware, a limited amount of test time, or the dynamic characteristics may be adequately described by a set of FRF measurements from a single reference location. The key to a successful modal test is understanding whether or not the set of measured FRFs are complete and accurate.

In single reference testing, it is assumed that all of the modes of interest are excited at the chosen reference location. If this is not the case, then those modes will not be excited, will not be represented in any of the FRF measurements, and will be "missed" in the analysis.

Another assumption that is made in single reference testing is that the structure obeys Maxwell's law of reciprocity. Unless measurements are obtained from other reference locations, the validity of both of these first two assumptions cannot be checked.

A third, and probably the most important assumption, of single reference testing is that the structure is linear and that its dynamics can be described by a set of second-order linear differential equations with constant coefficients. All of the modal parameter estimation methods are based on this model. If the experimental data does not match this model, poor modal parameter estimates will result. Many structures do behave in a linear manner and make this assumption valid. Others, typically large structures or those with many joints, do not behave linearly, especially when excited from one location.

It has already been shown [12], [13] that acquiring more than one row or column of the FRF matrix can improve the results of a modal test. The data can be acquired either **simultaneously** (more than one reference at a time) or **sequentially** (several different tests with different reference locations). Both methods will yield measurements that contain multiple estimates of the mode shapes for modes that are excited, i.e. that participate, at the reference locations. This redundant information can be used by multiple reference parameter estimation algorithms to obtain more accurate estimates of the modal parameters. Modes which do not participate at one reference may at another, and therefore should be correctly identified by the parameter estimation process. Repeated roots (more than one mode at the same frequency) can also be identified from this data using a multiple reference parameter estimation technique.

The major advantage of the sequential method of data acquisition is that it requires the minimum amount of test hardware. A two channel analyzer capable of an FRF measurement is all that is required. Because of this, however, the total test time will increase by a multiple of the number of different reference locations that are used. A major drawback of this method, though, is that it is no more effective than single reference testing in removing non-linearities from the measurements.

The simultaneous technique requires more test hardware than the sequential method; at least a 3 channel analyzer, extra shakers, extra signal generators, etc. More signal processing power is also needed in the data acquisition system to perform the matrix calculations necessary to obtain the FRFs. This approach is also faster than the sequential method in that once the test set up, the FRFs from more than one column are measured simultaneously.

The most important advantage of the simultaneous technique is its ability to improve the linearity of the FRF measurements. When more than one exciter is used, they are typically positioned on the structure so that they will effectively excite all the modes of interest, and the excitation energy will be distributed evenly over as much of the structure as possible.

Case 1: An Automobile Body-in-White

As a first case study of the multiple reference method, an automotive body-in-white was tested. The setup for this test is shown in Figure 1. The vehicle structure was mounted on four air rides to simulate a free-free boundary condition. Two exciters were attached to the lower front rails (points 1 and 2) in the vertical direction, and twenty biaxial accelerometers were positioned on the structure to measure responses.

Three separate tests were performed. Test #1 used a shaker at point 1 only. Test #2 used a shaker at point 2 only. Test #3 used shakers at points 1 and 2 simultaneously. All three tests used random-transient excitation to minimize the effects of leakage, and 50 power spectrum averages were taken for each measurement.

Modal tests are commonly preformed on car bodies to determine the first and second bending and torsion modes, for comparison with those from a finite element model. Without any prior knowledge of the test setup, one can usually look at the mode shapes obtained with a single exciter and clearly see where the exciter was positioned. The structure tends to look more compliant in the vicinity of the exciter. For example, the corner of the body where the shaker is attached will move significantly more than its symmetric counterpart. The reason for this is that this area of the structure is being driven past its linear range in order to develop a sufficient amount of measurable motion at the other end of the structure.

When the car body was tested with two shakers (references) simultaneously, the mode shapes where completely symmetrical, showed excellent correlation with the modes of the finite element model, and there was no evidence of distortion at the shaker locations. The more uniform energy distribution provided by multiple shakers, plus the signal processing of the simultaneous testing method, combine to remove the non-linear behavior from the FRF measurements.

Case 2: A Square Plate

To check the accuracy of the multiple reference curve fitting method described in the remainder of this paper, a simple square plate structure was tested. This test is interesting in that the theoretical natural frequencies and mode shapes can be calculated [1], [3], [4], and the plate contains repeated roots.
The plate structure was made of steel, 12 inches square, and .75 inches thick, as shown in Figure 4. For test purposes it was mounted on a piece of foam rubber to simulate free-free boundary conditions. A 4-channel data acquisition system was used to make FRF measurements. Three accelerometers where mounted on the plate normal to the surface, and were the reference locations. Excitation was provided by an impact hammer, and three FRFs were measured simultaneously. Data was collected for an impact at each of 144 points normal to the surface, for a total of 432 FRF measurements, which made up three rows of the FRF matrix.

CURVE FITTING BACKGROUND

Most analysis of the dynamics of structures is based upon the use of a set of linear second-order differential equations. For a structural model with n degrees-of-freedom, the equations can be written:

$$\left[M\right]\left\{x''(t)\right\}+\left[C\right]\left\{x'(t)\right\}+\left[K\right]\left\{x(t)\right\}=\left\{f(t)\right\} \qquad (n \text{ by } 1) \quad (1)$$

These equations are a statement of Newton's Second law involving all of the DOFs which are chosen for the model. The coefficient matrices, ($\left[M\right]$, $\left[C\right]$, & $\left[K\right]$), contain constants which represent the mass, damping, and stiffness properties of the structure, at least for the DOFs which are included in the model.

Since the equations of motion are linear, we can transform them into the Laplace domain without losing any information:

$$s^2\left[M\right]\left\{X(s)\right\}+s\left[C\right]\left\{X(s)\right\}+\left[K\right]\left\{X(s)\right\}=\left\{F(s)\right\}+\left\{ICs\right\} \qquad (n \text{ by } 1) \quad (2)$$

All of the physical properties of the structure are preserved on the left-hand side of the equations, while all of the applied forces and initial conditions (ICs) appear on the right-hand side. The initial conditions can be treated as a special form of the applied forces, and hence can be dropped from consideration in the following development without loss of generality.

To emphasize the three basic elements of any linear dynamical system, namely, the disturbances (or inputs), the responses (or outputs), and the physical system (signal filter), the equations of motion can be re-written:

$$\left[B(s)\right]\left\{X(s)\right\}=\left\{F(s)\right\} \qquad (n \text{ by } 1) \quad (3)$$

where:

$$\left[B(s)\right]=s^2\left[M\right]+s\left[C\right]+\left[K\right]= \text{the } \textbf{System Matrix} \qquad (n \text{ by } n) \quad (4)$$

Alternatively, the transformed equations of motion can be written:

$$\{X(s)\} = [H(s)]\{F(s)\} \qquad (n \text{ by } 1) \quad (5)$$

where $[H(s)]$ is called the **Transfer Matrix,** or the matrix of transfer functions.

Clearly, the System Matrix and the Transfer Matrix are inverses of one another. That is, they satisfy the equation:

$$[B(s)][H(s)] = [I] \qquad (n \text{ by } n) \quad (6)$$

where $[I]$ is the identity matrix.

The above equation is true for all values of the s-variable, and in particular for its values along the frequency axis ($j\omega$ -axis) in the s-plane. Hence, for all values of frequency, the following is true:

$$[B(j\omega)][H(j\omega)] = [I] \qquad (n \text{ by } n) \quad (7)$$

The matrix $[H(j\omega)]$ is called the Frequency Response Function Matrix, or simply the **FRF Matrix.** Expression (7) is particularly useful since it relates the physical properties of a structure, namely, its mass, damping, and stiffness properties, and FRFs, which can be readily measured on a structure with any modern-day multi-channel FFT analyzer.

MULTI-REFERENCE CURVE FITTING EQUATIONS

This curve fitting method is the outgrowth of some recent research that was first presented at IMAC III [2] in 1985. This original work was extended one year later to include the effects of initial conditions in the transformed equations [10]. This second implementation can potentially give better results, but the curve fitting equations still did not include the conjugate (negative frequency) poles associated with each mode. In a more recent paper, presented at IMAC IV [5], the curve fitting equations with negative frequency poles were presented. In our experience, the equations which include the negative frequency poles (shown below) have yielded more consistent answers than those which do not include these extra terms.

The inverse Laplace transform of the Transfer Matrix is a matrix of **Impulse Response Functions,** which we will denote by $[H(t)]$. For the cases of multiple input locations, $[H(t)]$ becomes a **rectangular matrix,** with the number of columns equal to the number of input DOFs, or references. Hence, if a set of measurements is made on a structure with n DOFs and r references then $[H(t)]$ would be an $(n \text{ by } r)$ matrix.

Writing out this matrix in terms of modal parameters:

$$[H(t)] = [U]\left[e^{pt}\right][L] + [U^*]\left[e^{p^* t}\right][L^*] \qquad (n \text{ by } r) \quad (8)$$

where $[U]$ is an $(n \text{ by } m)$ matrix of complex mode shapes (m = the number of modes). $\left[e^{pt}\right]$ is an $(m \text{ by } m)$ diagonal matrix, with each "p" in the diagonal elements corresponding to a mode's complex pole location (frequency and damping values).

$[L]$ is an $(m \text{ by } r)$ matrix of complex **modal participation factors.** These factors can be shown to be proportional to the mode shape values at the references.

Taking the time derivative of equation (8) yields:

$$[H'(t)] = [U]\left[pe^{pt}\right][L] + [U^*]\left[p^* e^{p^* t}\right][L^*] \qquad (n \text{ by } r) \quad (9)$$

At time $t = 0$, the initial displacement impulse response and its time derivative can then be written:

$$[H(t=0)] = [U][L] + [U^*][L^*] \qquad (n \text{ by } r) \quad (10)$$

$$[H'(t=0)] = [U][p][L] + [U^*][p^*][L^*] \qquad (n \text{ by } r) \quad (11)$$

Now, returning to the Laplace domain, the Transfer Matrix can also be written in terms of modal parameters:

$$[H(s)] = [U][T(s)] + [U^*][T^*(s)] \qquad (n \text{ by } r) \quad (12)$$

where: $$[T(s)] = [s-p]^{-1}[L] \qquad (n \text{ by } r) \quad (13)$$

The matrix $[s-p]$ is an $(m \text{ by } m)$ diagonal matrix, each term containing the pole location of a mode. Using equations (10), (11), and (12) above, we can now write the following identities:

$$s[H(s)] - [H(t=0)] = [U][p][T(s)] + [U^*][p^*][T^*(s)] \qquad (n \text{ by } r) \quad (14)$$

$$s^2[H(s)] - s[H(t=0)] - [H'(t=0)] = [U]\left[p^2\right][T(s)] + [U^*]\left[p^{*2}\right][T^*(s)] \qquad (n \text{ by } r) \quad (15)$$

Returning, for a moment, to the equations of motion (2), the modal properties are actually solutions to the homogeneous equations:

$$[U]\left[p^2\right] + [A_1][U][p] + [A_0][U] = [0] \qquad (n \text{ by } n) \quad (16)$$

where: $$[A_0] = [M]^{-1}[K] \qquad (n \text{ by } n) \quad (17)$$

$$[A_1] = [M]^{-1}[C] \qquad (n \text{ by } n) \quad (18)$$

The conjugate modal parameters ($\left[U^*\right]$, and $\left[p^*\right]$) are also solutions to the homogeneous equations. This can be written in a manner similar to equation (16).

Finally, we can pre-multiply equations (12), (14), and (15) by the matrices $\left[A_0\right]$, $\left[A_1\right]$, and $\left[I\right]$, and write the result as:

$$\begin{aligned}&\left[A_0\right]\left[H(s)\right]+\left[A_1\right]\left[s\left[H(s)\right]\right]-\left[H(t=0)\right]+\\&\left[I\right]\left[s^2\left[H(s)\right]\right]-s\left[H(t=0)\right]-\left[H'(t=0)\right]=\left[0\right]\end{aligned} \quad (n \text{ by } r) \quad (19)$$

Or, the above equation can be re-written:

$$\left[A_0\right]\left[H(s)\right]+\left[A_1\right]\left[s\left[H(s)\right]\right]+\left[B_0\right]+s\left[B_1\right]=-s^2\left[H(s)\right] \quad (20)$$

where:
$$\left[B_0\right]=-\left[A_1\right]\left[H(0)\right]-\left[H'(t=0)\right] \quad (n \text{ by } r) \quad (21)$$
$$\left[B_1\right]=-\left[H(t=0)\right]$$

Equation (20) is the **curve fitting equation,** and is valid for all values of the s-variable, in particular, those along the frequency axis $(s=j\omega)$. This equation is set up using measured FRF data $(\left[H(j\omega)\right])$, and solved for the unknown real-valued matrices, $\left[A_0\right]$, $\left[A_1\right]$, $\left[B_0\right]$, and $\left[B_1\right]$. The matrices $\left[A_0\right]$ and $\left[A_1\right]$ can then be used, together with equations (17) and (18), to recover the mass, stiffness, and damping matrices, as shown in a companion paper [11].

Alternatively, $\left[A_0\right]$ and $\left[A_1\right]$ can be used in equation (16) to solve for the modal parameters of the structure. Unfortunately, equation (20) must be solved for n DOFs in order to yield the $(n \text{ by } n)$ $\left[A_0\right]$ and $\left[A_1\right]$ matrices.

Practically speaking, we always work in experimental cases with a limited number of DOFs, and seek to identify an even smaller number of modes over a limited frequency range. This leads to a different formulation of the curve fitting equations.

GENERALIZED CURVE FITTING EQUATIONS

In this section, we will generalize the curve fitting equation (20) so that we can solve it for a limited number, (say m), of "generalized coordinates", or the so-called principle components. In doing this, we will find that some extra terms are needed in order to take into account the effects of the remaining "out-of-band" modes. This situation will always arise when m is smaller than n.

Let the full mode shape matrix $\left[U\right]$ be decomposed into two groups, the mode shapes of the modes within the frequency band of interest $\left[U_{ib}\right]$, and the shapes of the out-of-band modes $\left[U_{ob}\right]$. That is:

$$\left[U\right]=\left[U_{ib}\middle|U_{ob}\right] \quad (n \text{ by } n) \quad (22)$$

The $(n \text{ by } m)$ in-band mode shapes $\left[U_{ib}\right]$ can be expressed in terms of a linear transformation of its orthonormal principle coordinates $\left[V\right]$ $(n \text{ by } m)$, weighted by the generalized mode shape matrix $\left[v\right]$ $(m \text{ by } m)$, in the principle coordinates:

$$\left[U_{ib}\right]=\left[V\right]\left[v\right] \quad (n \text{ by } m) \quad (23)$$

where:
$$\left[V\right]^h\left[V\right]=\left[I\right] \quad (m \text{ by } m) \quad (24)$$

$\left[I\right]$= the identity matrix

h – denotes the Hermitean (transposed conjugate)

The subset of equation (16) for only the in-band modes can be written:

$$\left[U_{ib}\right]\left[p_{ib}^2\right]+\left[A_1\right]\left[U_{ib}\right]\left[p_{ib}\right]+\left[A_0\right]\left[U_{ib}\right]=\left[0\right] \quad (n \text{ by } m) \quad (25)$$

Premultiplying equation (25) by the Hermitean of the mode shapes of the principle components gives:

$$\left[I\right]\left[v\right]\left[p_{ib}^2\right]+\left[a_1\right]\left[v\right]\left[p_{ib}\right]+\left[a_0\right]\left[v\right]=\left[0\right] \quad (m \text{ by } m) \quad (26)$$

where:
$$\left[a_0\right]=\left[V\right]^h\left[A_0\right]\left[V\right] \quad (m \text{ by } m) \quad (27)$$

$$\left[a_1\right]=\left[V\right]^h\left[A_1\right]\left[V\right] \quad (m \text{ by } m) \quad (28)$$

Equation (26), a much reduced-in-size version of equation (16), can be used to solve for the in-band poles (modal frequencies and damping), and generalized mode shape matrix in principle coordinates $\left[v\right]$, once the matrices $\left[a_0\right]$ and $\left[a_1\right]$ are known. Mode shapes in physical coordinates can be recovered using equation (23), once the generalized mode shapes $\left[v\right]$, and the matrix of principle coordinates $\left[V\right]$ are known.

The generalized curve fitting equation is derived by:

(1) premultiplying equation (12) by the matrix $\left[V\right]^h$

(2) premultiplying equation (14) by the matrix product $\left[a_0\right]\left[V\right]^h$

(3) premultiplying equation (15) by the matrix product

$$[a_1][V]^h$$

(4) summing together the results from steps (1), (2), and (3) above

(5) applying equation (26) to the in-band mode portion of the resulting equation.

The following curve fitting equation will result:

$$\begin{aligned}&[a_0][h(s)]+[a_1][s[h(s)]-[h(t=0]]\\&+[I][s^2[h(s)]-s[h(t=0)]-[h'(t=0)]]\\&=[V]^h[[U_{ob}][T(s)_{ob}]+[U^*_{ob}][T(s)^*_{ob}]]\\&+[V]^h[[U_{ob}][p_{ob}][T(s)_{ob}]+[U^*_{ob}][p^*_{ob}][T(s)^*_{ob}]]\\&+[V]^h[[U_{ob}][p^2_{ob}][T(s)_{ob}]+[U^*_{ob}][p^{*2}_{ob}][T(s)^*_{ob}]]\end{aligned}$$

$$(m \text{ by } r) \quad (29)$$

where:
$$[h(s)]=[V]^h[H(s)] \qquad (m \text{ by } r) \quad (30)$$
$$[h(t=0)]=[V]^h[H(t=0)] \qquad (m \text{ by } r) \quad (31)$$
$$[h'(t=0)]=[V]^h[H'(t=0)] \qquad (m \text{ by } r) \quad (32)$$
$$[T(s)_{ob}]=[s-p_{ob}]^{-1}[L_{ob}] \qquad ((m-n) \text{ by } r) \quad (33)$$

Equation (29) can be simplified:

$$\begin{aligned}&[a_0][h(s)]+[a_1][s[h(s)]]+[b_0]+s[b_1]\\&+[c_0][s-p_{ob}]^{-1}+[c_1][s-p^*_{ob}]^{-1}=-s^2[h(s)]\end{aligned}$$
$$(m \text{ by } r) \quad (34)$$

where $[b_0]$, $[b_1]$, $[c_0]$, and $[c_1]$ are all constant matrices, independent of the s-variable.

Equation (34) is the generalized curve fitting equation, which will yield solutions for a selected number of in-band modes, and requires a reasonable number of FRF measurements defined over a limited frequency range. (Note that this equation is also valid for all values of the s-variable, in particular those along the $j\omega$ axis, where the FRF measurement data $[H(j\omega)]$, is defined). The curve fitting equation is solved for the unknown matrices, $[a_0]$, $[a_1]$, $[b_0]$, $[b_1]$, $[c_0]$, and $[c_1]$. Typically, a least squared error form of the curve fitting equation is solved, instead of equation (34) itself. This allows more flexibility in terms of the amount of FRF data used in the solution procedure.

For practical applications, only a small number, say 2, of out-of-band modes, immediately below and above the frequency band of interest, are needed in the solution equations to compensate for the effects of all of the out-of-band modes. If the out-of-band mode terms are left out of the equations, however, significant errors can occur in the estimates of the modal parameters of the in-band modes, especially the damping estimates. (This will be illustrated later).

If the principle components are assumed to be real-valued, then the matrices $[a_0]$, $[a_1]$, $[b_0]$, and $[b_1]$ will be real-valued. This assumption, however, still allows complex mode shapes to be calculated, since the matrix $[v]$ is complex-valued in general. Furthermore, if the damping of the out-of-band modes is neglected, equation (34) can be rewritten:

$$\begin{aligned}&[a_0][h(s)]+[a_1][s[h(s)]]+[b_0]+s[b_1]\\&+[c_2][s^2-p^2_{ob}]^{-1}+[c_3]s[s^2-p^2_{ob}]^{-1}=-s^2[h(s)]\end{aligned}$$
$$(m \text{ by } r) \quad (35)$$

where $[c_2]$ and $[c_3]$ are unknown real-valued matrices.

Equation (35) is the generalized form of the curve fitting equation which contains additional terms to compensate for the effects of out-of-band modes. FRF measurement data, transformed into principle coordinates $[h(s)]$, is used in the equations, and they are solved, in a least squared error form, for six unknown coefficient matrices. The unknown matrices $[a_0]$ and $[a_1]$ contain the structural properties, in principle coordinates. The unknown matrices $[b_0]$ and $[b_1]$ contain the effects of initial conditions, while the matrices $[c_2]$ and $[c_3]$ account for the effects of the out-of-band modes.

PRINCIPLE COMPONENT REDUCTION

A proper selection of the principle components of the mode shapes of interest can be critical to the successful use of this curve fitting method. The best source for determining the principle components is the FRF measurements themselves, which contain a weighted summation of the mode shapes. Furthermore, the FRF data in the vicinity of each of the resonance peaks within the frequency band of interest contains the highest signal-to-noise ratio, and is the most suitable for computing the principle components.

To find the principle components, the FRF measurements are first written as a matrix triple product:

$$[H]=[V][d][W] \qquad (n \text{ by } (r\times f)) \quad (36)$$

where: $[V]^h[V]=[I]$ $\quad ((r\times f)\text{ by }(r\times f))$ (37)

$[d]$ = diagonal matrix $\quad ((r\times f)\text{ by }(r\times f))$ (38)

$[W]^h=[W]^{-1}$ $\quad ((r\times f)\text{ by }(r\times f))$ (39)

The matrix $[V]$ is calculated by solving for the eigenvectors of the matrix:

$$[H][H]^h=[V][d][d]^h[V]^h \qquad (n\text{ by }n) \quad (40)$$

The number of "significantly" non-zero eigenvalues (diagonals of the matrix $[d][d]^h$) gives an indication of the rank of the above matrix, and also indicates the number of significant principle components in the FRF measurements. This number is also used as the indication of the number of modes represented by the FRF data. The columns of the $[V]$ matrix which correspond to the "significant" eigenvalues are taken as the principle components.

$[V]$, $(n\text{ by }m)$, is all that is needed to set up the curve fitting equations. For the usual applications of light damping, the principle components can be taken as real numbers, which means that only the real part of the left-hand side of equation (40) is used to calculate the real matrix $[V]$.

Alternatively, if the number of measurement DOFs, n, is very large, the following equation can be solved for the matrix $[W]$.

$$[H]^h[H]=[W]^h[d]^h[d][W] \qquad ((r\times f)\text{ by }(r\times f)) \quad (41)$$

$[W]$ can then be used in equation (36) to solve for $[V]$.

ANALYTICAL VERIFICATION: A 5-DOF MODEL

This new curve fitting method was first verified by using it on a 5-DOF analytical model, shown in Figure 2, where the correct values of its modal parameters were known beforehand. The five point masses of the model are connected together with linear springs and dampers, and the fifth mass is also connected to ground.

The modal frequencies, damping, and mode shapes for the structure are shown in Table 1. FRF measurements were then synthesized, using this modal data, for three reference points; masses 1, 2, and 3. An example FRF is shown in Figure 3.

First, the FRF measurements were curve fit using a least squared error version of equation (20), to obtain estimates of the parameters for all five modes. During the curve fitting process, FRF data from all five response DOFs, all three references, and in 2 Hz frequency bands surrounding each resonance peak, was used. The results were found to be identical to the values listed in Table 1. This can be expected when all of the modes of a system are simultaneously identified using FRF data that contains no residual effects of additional modes.

Effects of Out-of-Band Modes

In most modal test situations we collect FRF measurements that contain the residual effects of out-of-band modes. It will be demonstrated next that the improved curve fitting equation (35) is required in order to take into account these out-of-band effects.

The synthesized FRF measurements of the 5-DOF system were curve fit to find the parameters for the second and third modes only. Again, the FRF data for all five response DOFs, all three references, and in 2 Hz frequency bands around the resonance peaks of the second and third modes was used.

Errors resulting from the use of curve fitting equation (20), which contains no compensation for the effects of the out-of-band modes, are shown in Table 2. Errors as great as 33.9% (for the damping of the third mode) occurred.

The errors resulting from the use of the generalized curve fitting equations (35) are also shown in Table 2. In this case, all errors are less than 0.32%. In the use of equation (35), approximate frequencies of the first and fourth modes, i.e. 9.0 and 46.0 Hz, with no damping, were used in the out-of-band terms. As the results indicate, the inclusion of two out-of-band terms adequately compensated for the effects of all three actual out-of-band modes.

Figure 2. The Five Degree-of-Freedom Model

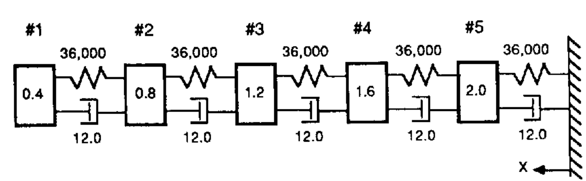
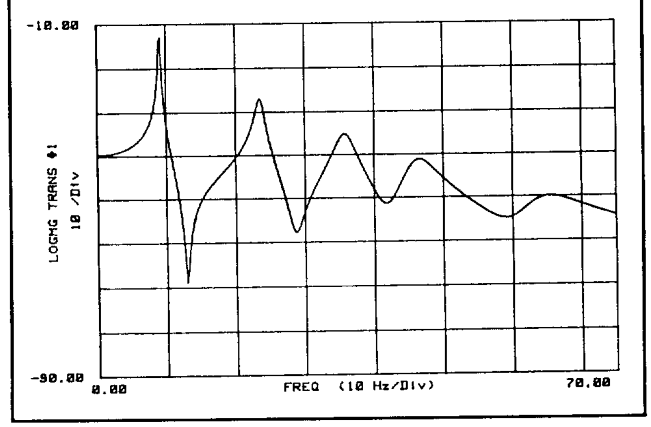


Table 1. Modal Parameters of the Five DOF Model

Mode Number		1	2	3	4	5
Frequency (Hz)		9.11	23.56	35.81	46.14	63.89
Damping (%)		0.95	2.47	3.75	4.83	6.69
Mode Shapes	DOF#1	0.580	0.635	0.593	0.623	1.011
	DOF#2	0.559	0.480	0.259	0.041	- 0.799
	DOF#3	0.497	0.092	- 0.366	- 0.617	0.252
	DOF#4	0.380	- 0.364	- 0.373	0.453	- 0.050
	DOF#5	0.209	- 0.465	0.459	- 0.170	0.007

Figure 3. Synthesized FRF from 5-DOF Model



EXPERIMENTAL VERIFICATION: A SQUARE PLATE MODEL

In this section the ability of the curve fitting method to correctly identify repeated roots (or modes) is examined. The curve fitter is used on some FRF measurements from a square plate. Because of the symmetry of its geometry, a square plate will often have repeated modes, which cannot be resolved correctly using a single reference curve fitting method. If there are two repeated roots, then at least two references (rows or columns) of the FRF matrix must be used in order to identify the modal parameters.

The square plate, shown in Figure 4, was tested in a free-free condition using impact testing. The structure was impacted at 144 points, in the normal direction to the surface, and three different reference (or response) points were used, resulting in a total of 432 measurements. The measurements were made over a frequency range from DC (zero frequency) to 2,000 Hz, with 5 Hz resolution between frequency lines. An example of a driving point FRF measurement is shown in Figure 5.

Modal Frequencies

From visual inspection, the measurements only contain evidence of four flexual modes below 800 Hz. (The peak at 25 Hz is due to the rigid body modes). But, the curve fitter found five modes. The frequency and damping estimates are listed in Table 3. Notice that even though the frequency resolution of the FRF data was 5 Hz, the curve fitter found fourth and fifth modes which were about 1 Hz apart, at 577.2 and 578.5 Hz.

Table 2. Out-of-Band Mode Effects: 5-DOF Model

			Without Out-of-Band Modes		With Two Out-of-Band Modes	
		Exact Solution	Result	% Error	Result	% Error
Frequency	Mode 2	23.56	23.61	0.2	23.56	0.0
(Hz)	Mode 3	35.81	35.85	0.1	35.82	0.0
Damping	Mode 2	2.47	2.76	11.7	2.47	0.0
(%)	Mode 3	3.75	5.02	33.9	3.76	0.3

Analytical derivations of the frequencies of the modes of a square plate with free-free boundary conditions are readily available in the literature [1], [3], [4]. The analytical frequencies are shown in Table 4. Also shown are comparisons of the analytical and test frequencies, with and without adjustments for the rigid body modes at 25 Hz. The comparisons show very close agreement between the analytical and test results, especially for the adjusted test frequencies. No further attempts were made to pin down the detailed differences between the analytical and test models. It is clear, however, that the repeated modes found were not “computational” modes.

Mode Shapes

The existence of repeated modes is further confirmed by an examination of the mode shapes. As shown in Figures 6d and 6e, the mode shapes of the fourth and fifth modes are mirror images of one another. The mode shapes in Figure 6 agree closely (by a nodal line comparison) with those reported elsewhere in the literature [1].

Modal Damping

The damping of the test structure is more difficult to verify by comparison with analytical results. The damping estimates in Table 3 do, however, agree with the half power point widths of the resonance peaks in the FRF data. If equation (20) was also used to curve fit the square plate data. The results are shown in Table 6. These estimates are judged to be erroneous because of the uncharacteristic disparity of the damping values.

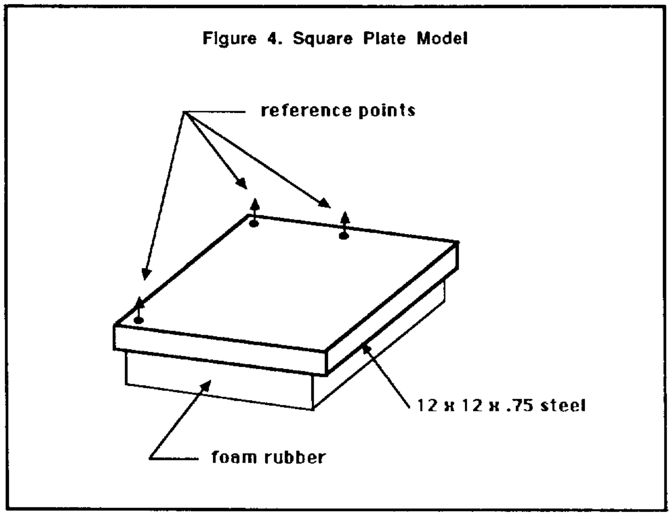


Table 3. Poles of the Square Plate from Test Data

	Frequency (Hz)	Damping (%)	Damping (Hz)
Mode 1	241.5	1.90	4.58
Mode 2	340.1	0.86	2.92
Mode 3	379.3	0.85	3.22
Mode 4	577.2	0370	4.03
Mode 5	578.5	0.77	4.45

Rank Indicator

Clearly one of the most useful functions of this curve fitting algorithm is its strong indication of the number of modes represented in the FRF data, by its calculation of the rank of the FRF matrix. As shown in Table 5, the rank indication clearly shows that there are five modes in the data, even though FRF data from the vicinity of only four resonance peaks was used in the principle component calculation.

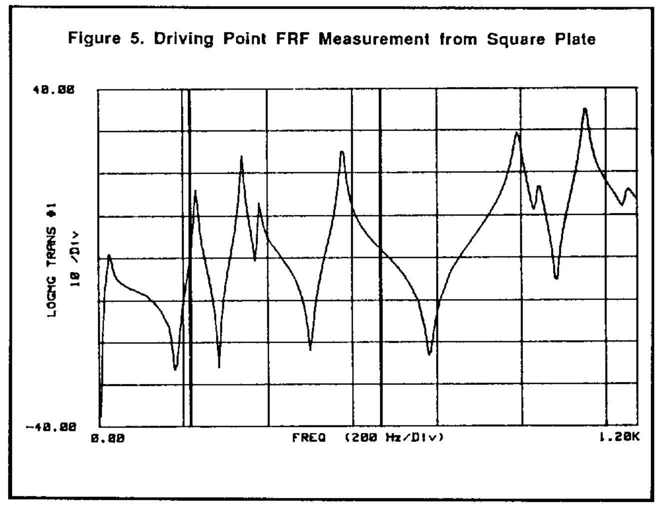
The rank indicator works best when used with only the FRF data from around the resonance peaks, since this data contains the strongest signal-to-noise ratio of the principle components, or mode shapes, of interest. The rank indicator, when used properly, can free the user of the sometimes difficult task of determining how many modes are actually in the data, a prerequisite to using any type curve fitter. This is an even bigger advantage in cases of closely coupled modes, and/or poor frequency resolution.

Table 4. Comparison of Square Plate Frequencies

	FREQUENCY RATIO		
	Analytical*	Test* Adjusted	Test* Unadjusted
Mode 1	1.00	1.00	1.00
Mode 2	1.47	1.46	1.41
Mode 3	1.81	1.64	1.57
Mode 4	2.60	2.55	2.39
Mode 5	2.60	2.56	2.40

$$\text{RATIO} = \frac{\text{Frequency}\,(k)}{\text{Frequency}\,(1)}$$

$$\text{RATIO} = \frac{\text{Frequency}\,(k) - \text{Frequency}\,(\text{rigid})}{\text{Frequency}\,(1) - \text{Frequency}\,(\text{rigid})}$$



CONCLUSIONS

We have addressed some cases where multiple reference modal testing and curve fitting offer distinct advantages over more conventional approaches. Multiple shaker testing can more effectively remove non-linearities when testing large structures, and yield a more consistent set of modal parameters. Multiple reference impact testing more effectively treats local modes, structures with uni-directional modes, and reduces the likelihood of missed modes.

We have extended the usefulness of a frequency-domain multi-reference curve fitter by adding terms to the solution equations which account for out-of-band modes. In the cases given here, this residual compensation clearly gave improved results, especially the modal damping estimates which are typically the most difficult parameters to estimate accurately.

Finally, we showed that multi-reference curve fitting can correctly identify modes which are repeated roots, a case which cannot be handled either theoretically or practically with single reference methods.

Table 5. Rank Indicator for the Square Plate

```
***      CRITERIA TO JUDGE RANK      ***
****************************************
COMP NO 1    VAL = .250409 +06   @***                       @
COMP NO 2    VAL = .169710 +06   @**                        @
COMP NO 3    VAL = .125909 +06   @ ***********              @
COMP NO 4    VAL = .268754 +05   @**                        @
COMP NO 5    VAL = .240035 +05   @************************@
COMP NO 6    VAL = .207035 +04   @*********                 @
COMP NO 7    VAL = .567341 +03   @**                        @
COMP NO 8    VAL = .432370 +03   @**                        @
COMP NO 9    VAL = .401080 +03   @**                        @
COMP NO 10   VAL = .295200 +03   @****                      @
COMP NO 11   VAL = .149435 +03   @**                        @
COMP NO 12   VAL = .127768 +03   @*****                     @
COMP NO 13   VAL = .564410 +02   @**                        @
COMP NO 14   VAL = 3489944 +02   @***                       @
COMP NO 15   VAL = .327520 +02   @**                        @
COMP NO 16   VAL = .244444 +02   @**                        @
COMP NO 17   VAL = .186361 +02   @**                        @
COMP NO 18   VAL = .138792 +02   @**                        @
COMP NO 19   VAL = .126893 +02   @***                       @
```

Figure 6. Mode Shapes of Square Plate Showing Nodelines

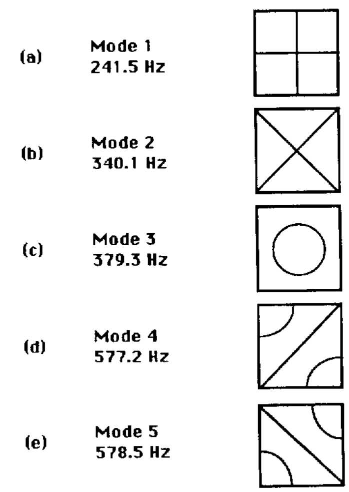


Table 6. Out-of-Band Mode Effects: Square Plate Model

		With two Out-of-Band Modes	Without Out-of-Band Modes	Percent Error
	Mode 1	1.90	5.56	193%
	Mode 2	0.86	1.24	44%
Damping	Mode 3	0.85	0.99	16%
(%)	Mode 4	0.70	0.33	52%
	Mode 5	0.77	0.39	49%
	Mode 1	241.5	216.6	10.3%
	Mode2	340.1	340.1	0.0%
Frequency	Mode 3	379.3	380 .2	0.2%
(Hz)	Mode 4	577.2	577.7	0.1%
	Mode 5	578.5	579.1	0.1%

Multiple reference modal testing and curve fitting is more time consuming and more costly to implement than the more popular single reference methods, and for the majority of situations, their use is probably not warranted. Nevertheless, it's reassuring to know that multiple reference methods have already been developed to the point where they can be used today in a laboratory environment, to more accurately characterize the dynamic properties of structures.

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