Simultaneous Structural Dynamics Modification (S^{2}DM)

by

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ABSTRACT

Structural Dynamics Modification has become a practical tool in the last few years for improving the engineering designs of mechanical systems. It provides a very quick and inexpensive approach for investigating the effects of design modifications to a structure, thus eliminating the need for costly prototype fabrication and testing. It also has the unique advantage of working directly with data obtained either from a modal test, or from a finite element model of the structure.

Once the modal properties of a structure are known, SDM, which is based on the eigenvalue modification technique [1], can be used to predict the dynamic effects of certain kinds of "local" design modifications. These modifications are typically expressed in the form of point mass, scalar spring, and scalar damper additions or removals at or between the existing test points of the structure. This technique is very efficient computationally, since the size of the eigenvalue problem to be solved is typically orders of magnitude less than an eigenvalue problem in special coordinates, e.g. finding the eigenvalues and eigenvectors of a finite element model.

While the application of scalar mass, spring, or damper modifications can be handled very efficiently with the currently available numerical schemes for SDM, more general matrix type modifications, which represent perhaps hundreds of scalar element modifications, are more desirable for most practical applications.

In this paper, the local modification method is formulated so that general matrix modifications can be made with it. Then, a case study is presented to demonstrate its capability, accuracy, and computational efficiency. This new implementation of SDM is called S2DM since it yields in one eigensolution the same result as repeated applications of the SDM method.

NOMENCLATURE

n = number of DOFs of the dynamic model
m = number of modes
t = time variable
s = Laplace variable

$[M]$ = mass matrix $(n\,by\,n)$
$[C]$ = damping matrix $(n\,by\,n)$
$[K]$ = stiffness matrix $(n\,by\,n)$

$[m]$ = modal mass matrix $(2m\,by\,2m)$
$[c]$ = modal damping matrix $(2m\,by\,2m)$
$[k]$ = modal stiffness matrix $(2m\,by\,2m)$

$\{x(t)\}$ = vector of displacements $(n\,by\,1)$
$\{z(t)\}$ = vector of modal displacements $(2m\,by\,1)$
$\{x'(t)\}$ = vector of velocities $(n\,by\,1)$
$\{x''(t)\}$ = vector of accelerations $(n\,by\,1)$
$\{f(t)\}$ = vector of externally applied forces $(n\,by\,1)$

$\{ICs\}$ = vector of initial condition terms $(n\,by\,1)$
$\{X(s)\}$ = Laplace transforms of displacements $(n\,by\,1)$
$\{Z(s)\}$ = Laplace transforms of modal displacements $(2m\,by\,1)$

$[B(s)]$ = system matrix $(n\,by\,n)$
$[U]$ = matrix of mode shapes $(n\,by\,2m)$

p_k = pole location for the k^{th} mode = $-\sigma_k + j\omega_k$

σ_k = damping of the k^{th} mode

ω_k = damped natural frequency of the k^{th} mode, $k = 1,\ldots,m$

INTRODUCTION

A variety of numerical methods have been developed over the years which use only the eigenvalues and eigenvectors, (i.e. the modal properties) of a structure, and predict the structure's new modal properties after a physical change has been made to it. These same techniques can also be used for substructuring; that is, taking the modal properties of a number of subsystems and deriving the modal properties of the overall system after the subsystems have been connected together with spring and damper elements. Among the more traditional methods for performing these modifications are **modal synthesis**, the **Lagrange multiplier** method, and **diakoptics.** More recently, the **local eigenvalue modification** technique has been developed, primarily through the work of Weissenburger, Pomazal, Hallquist, and Snyder. Reference [2] provides a summary of these developments.

All of the above work was done primarily with analytical data. The local eigenvalue modification method was developed for use with finite element models. The primary objective was to provide a more efficient means for investigating physical changes to a structure, by making modifications to its dynamic model and then solving for a new eigensolution without solving an eigenvalue problem in physical coordinates.

In 1978, we at SMS began using the local eigenvalue modification method with modal data which was derived directly from a modal test. This implementation was made available to the public in the form of commercial software which could be used in a laboratory test system. The computational efficiency of this method made it very attractive for implementation in a desktop computer system which could

be used in a laboratory. importantly, however, was the fact that it gave reasonably accurate results, even with laboratory derived modal data, and with only a relatively small number of modes represented in the data base.

Some of the effects of using a small number of modes in the data base, referred to as modal truncation, were presented in [3]. Other comparative results were presented in [1]. A key advantage of local eigenvalue modification is that it requires only modal data to characterize the dynamics of the structure, and it directly provides a new set of modal data for the modified structure. Hence, a series of more complex modifications can be performed by applying modifications, one after another, to the modal data which results from each previous modification.

As will be shown in the following theoretical development, the computational efficiency of local eigenvalue modification comes from the fact that scalar modifications (point masses, linear springs, and linear dampers) only require the solution of a one dimensional eigenvalue problem in order to determine the modified modal properties. Hence computational speed is independent of the number of physical DOFs in the model, and very large models can be handled as efficiently as small ones.

Although this technique is very advantageous when only a few scalar modifications are needed, its computational advantage, both in terms of speed and accuracy, diminishes as the number of modifications to be made increases.

The fundamental process of structural modifications is the solution of an eigenvalue problem. Using a straightforward approach, if the mass, damping, and stiffness properties of the structure are known, then the modal properties of the modified structure would be found by first making the appropriate changes to the mass, damping, and stiffness coefficient matrices of the equations of motion, and then solving for the eigenvalues and eigenvectors of the modified equations. This would amount to solving an eigenvalue problem in physical coordinates. For example, if there were 1000 DOFs in the dynamic model, the eigensolution would require the manipulation of matrices of size 1000 by 1000.

What makes the SDM method efficient is that it solves the eigenvalue problem in modification space instead of physical space. Hence, to add a single stiffener to a structure which is modeled with 1000 DOFs using SDM, only a scalar (1 by 1) eigenvalue problem is solved instead of a 1000 by 1000 problem.

An alternative approach is to solve the modified equations of motion in modal space, which is still more efficient than solving the problem in physical space. Hence, if the eigenvalue problem is formulated in modal space and the dynamics of the structure are represented by the parameters of 10 modes of vibration, then the modes of the modified structure can be found by solving a 10 by 10 problem instead of a 1000 by 1000 problem.

The advantage of modal space over modification space is that by formulating the problem in modal space, the size of the problem is independent of the number of scalar modifications made to the structure. Therefore, many modifications can be modeled simultaneously in modal space, and the eigenvalue problem size, and therefore the execution time, does not increase.

BACKGROUND THEORY

The local eigenvalue modification process begins with a dynamic model of the unmodified structure. This model can be represented either in terms of the mass, damping and stiffness properties of the structure, or in terms of its modal properties: frequencies, damping, and mode shapes. The modal data can be obtained either from a modal test of the structure, or from an eigensolution of the differential equations of motion, which are typically generated with finite element modeling techniques.

Mass, stiffness, and damping modifications are made to the structure by making additions to, or subtractions from, the mass, stiffness, or damping coefficient matrices of the differential equations of motion.

The equations of motion of the unmodified structure are written as a set of linear second-order differential equations. For a structural model with n-degrees of freedom, they are written as:

$$[M]\{x''(t)\}+[C]\{x'(t)\}+[K]\{x(t)\}=\{f(t)\} \quad (n\,by\,1)\,(1)$$

Similarly, the equations of motion of the modified structure are written as:

$$[M+\Delta M]\{x''(t)\}+[C+\Delta C]\{x'(t)\}+[K+\Delta K]\{x(t)\}=\{f(t)\}$$

$$(n\,by\,1)\,(2)$$

where the matrices $[\Delta M]$, $[\Delta C]$, $[\Delta K]$ contain the mass, damping, and stiffness modifications, respectively, of the modified structure.

Transformed Equations of Motion: Since the equations of motion are linear, they can be transformed to the frequency domain using the Laplace transform without losing any information. The equations then take the form:

$$s^2[M]\{X(s)\}+s[C]\{X(s)\}+[K]\{X(s)\}=\{F(s)\}+\{ICs\}$$

$$(n\,by\,1)\,(3)$$

All of the physical properties of the structure are preserved in the left-hand side of the equations, while the applied forces and initial conditions $\{ICs\}$ are contained on the right-hand side. The initial conditions can be treated as a special form of the applied forces, and hence can be dropped from consideration in the following development without loss of generality.

To emphasize the three basic elements of any linear dynamic system, namely, the externally applied disturbances (inputs), the responses (outputs), and the physical system (linear filter), the transformed equations of motion can be rewritten as:

$$[B(s)]\{X(s)\}=\{F(s)\} \quad (n\,by\,1)\,(4)$$

where:

$$[B(s)]=s^2[M]+s[C]+[K] \quad (n\,by\,n)\,(5)$$

$[B(s)]$ is defined as the system matrix.

Modal Coordinates: The modal parameters of a structure are actually the solutions to the homogeneous equations of motion. That is, when $\{F(s)\}=\{0\}$ the solutions to equations (4) are complex valued eigenvalues and eigenvectors. The complex eigenvalues occur in conjugate pairs, (p_k, p_k^*), and are solutions to the determinant equation:

$$\det([B(s)])=0 \quad (1\,by\,1)\,(6)$$

The eigenvalues, or **poles** of the system, can be written:

$$p_k=\sigma_k+j\omega_k, \qquad k=1,\ldots,m$$

$$p_k^*=\sigma_k-j\omega_k, \qquad k=1,\ldots,m$$

Each eigenvalue has an eigenvector corresponding to it, and hence the eigenvectors also occur in conjugate pairs $(\{u_k\},\{u_k^*\})$ as solutions to the equations:

$$[B(p_k)]\{u_k\}=\{0\}, \qquad k=1,\ldots,m \qquad (n\,by\,1)$$
(7)

$$[B(p_k^*)]\{u_k^*\}=\{0\}, \qquad k=1,\ldots,m \quad (n\,by\,1)\,(8)$$

The eigenvectors, or **mode shapes,** can be assembled into a matrix:

$$[U]=[\{u_1\},\{u_2\},\ldots,\{u_m\}\ \{u_1^*\},\{u_2^*\},\ldots\{u_m^*\}] \quad (n\,by\,2m)\,(9)$$

Using the mode shape matrix, the motion of the structure can be represented in modal coordinates as:

$$\{x(t)\}=[U]\{z(t)\} \quad (n\,by\,1)\,(10)$$

Applying this transformation to equations (4) gives:

$$[s^2[M][U]+s[C][U]+[K][U]]\{Z(s)\}=\{F(s)\} \quad (n\,by\,1)\,(11)$$

Then, premultiplying equation (11) by the transposed conjugate of the mode shape matrix $([U]^t)$ gives:

$$[s^2[U]^t[M][U]+s[U]^t[C][U]+[U]^t[K][U]]\{Z(s)\}=[U]^t\{F(s)\}$$

$$(2m\,by\,2m)\,(12)$$

We can now define three new matrices:

Modal Mass = $[m] = [U]^t [M][U]$ $(2m\,by\,2m)$ (13)

Modal Damping = $[c] = [U]^t [C][U]$ $(2m\,by\,2m)$ (14)

Modal Stiffness = $[k] = [U]^t [K][U]$ $(2m\,by\,2m)$ (15)

The equations of motion **in modal coordinates** now become:

$$\left[s^2[m] + s[c] + [k]\right]\{Z(s)\} = [U]^t \{F(s)\} \qquad (2m\,by\,2m) \quad (16)$$

Damping Assumptions: So far, no assumptions have been made about damping other than the fact that it can be modeled with a linear model. If no further assumptions are made, then the modal mass, damping, and stiffness matrices are, in general, full (non-diagonal) matrices. This case is referred to as **effective linear,** or **non-proportional** damping.

If, however, it is assumed that the structure has **no damping** $([C] = [0])$, then it can be shown that the equations of motion (16) are uncoupled; that is, the modal mass and stiffness matrices are diagonal matrices. Alternatively, if it is assumed that the damping matrix is **proportional to the mass or stiffness matrices,** ($[C] = \alpha[M] + \beta[K]$, α, β are proportionality constants), then the equations of motion (16) are again uncoupled, and the modal mass, damping, and stiffness matrices are diagonal matrices.

Unfortunately, neither of the above damping assumptions applies very well to real structures. Real structures always have some amount of damping, and there are no physical reasons for assuming that damping is proportional to mass or stiffness.

A better assumption, and one which will yield an approximation to the uncoupled equations, is to assume that **the damping forces are significantly less than the Inertial (mass) or the restoring (stiffness) forces** on the structure. In other words, an assumption is made that the structure is **lightly damped.**

Most structures which exhibit resonance conditions can be considered as lightly damped structures. Consequently, their modes are also lightly damped. Structures with modal damping of **10 percent of critical or less can be** considered as lightly damped.

If light damping is assumed, then it can also be shown that the modal mass, damping, and stiffness matrices approximate diagonal matrices. Furthermore, the mode shapes can be shown to be approximately real valued so that the *2m* equations become redundant, and can be replaced with *m* equations, one corresponding to each mode.

All of the above cases of damping can be summarized as follows:

Damping	Mode Shapes	Modal Matrices
Non-Proportional	Complex	Non-Diagonal $(2m\,by\,2m)$
None	Real	Diagonal $(m\,by\,m)$
Proportional	Real	Diagonal $(m\,by\,m)$
Light	Almost Real	Almost Diagonal $(m\,by\,m)$

If the mode shapes, which are eigenvectors, are scaled so that the modal mass matrix diagonal elements are unity, then the modal mass matrix becomes an identity matrix, and the equations of motion become:

$$\left[s^2[I] + s[2\sigma] + [\Omega^2]\right]\{Z(s)\} = [U]^t \{F(s)\} \qquad (m\,by\,1) \quad (17)$$

where:

$[I]$ = identity matrix $(m\,by\,m)$

$[2\sigma]$ = diagonal modal damping matrix $(m\,by\,m)$

$[\Omega^2]$ = diagonal modal frequency matrix $(m\,by\,m)$

$[\Omega^2] = [\sigma^2 + \omega^2]$

From equation (17) it is clear that the entire dynamics of the unmodified structure can be represented by modal parameters: frequencies, damping, and mode shapes scaled to unit modal masses.

The equations of motion for the **modified structure,** transformed to modal coordinates, can be written in a similar manner as:

$$\left[s^2[m] + s[c] + [k]\right]\{Z(s)\} = [U]^t \{F(s)\} \quad (m\,by\,1) \quad (18)$$

where:

$$[m] = [I] + [U]^t [\Delta M][U] \qquad (m\,by\,m) \quad (19)$$

$$[c] = [2\sigma] + [U]^t [\Delta C][U] \qquad (m\,by\,m) \quad (20)$$

$$[k] = [\Omega^2] + [U]^t [\Delta K][U] \qquad (m\,by\,m) \quad (21)$$

In this case, the mode shape matrix is of dimension $(n\,by\,m)$ since the mode shapes are approximately real valued.

The homogeneous form of equation (18) must be solved to find the modal properties of the modified structure. Using the approach of Hallquist, et al [2], an additional transformation of the modification matrices $\left[\Delta M\right]$, $\left[\Delta C\right]$, $\left[\Delta K\right]$ is made which results in a reformulation of the eigenvalue problem in modification space. For a single modification, this problem becomes a scalar eigenvalue problem, which can be solved quickly and efficiently. The drawback to making one modification at a time, however, is that if a large number of modifications are to be made, computation time and computational errors can become significant.

The approach taken here is to solve the homogeneous form of equation (18) directly. This is still a relatively small eigenvalue problem (# **of modes** by **# of modes)** which only needs to be solved once for as many modifications as desired.

A Typical Application

SDM is very useful for investigating various mounting configurations of structures. In cases like these, the rigid body modes are used along with the free-free flexible body modes of the structure to model its free body dynamics. Then the structure is “mounted” by connecting it to ground (or perhaps some other elastic structure) with springs and dampers.

In the example considered here, a flat aluminum plate was tested to obtain its free-free flexible body modes, and its rigid body modes were synthesized and added to the flexible modes to form the unmodified structural model.

Assuming that only motion in the vertical (Z) direction is of interest, then only three of the six rigid body modes are needed; linear motion in the Z-direction, rotation about the X-axis, and rotation about the Y-axis.

The frequencies and damping of the modal model, which were used as input to SDM, are listed below. Typical mode shapes for several of the flexible modes are also shown below.

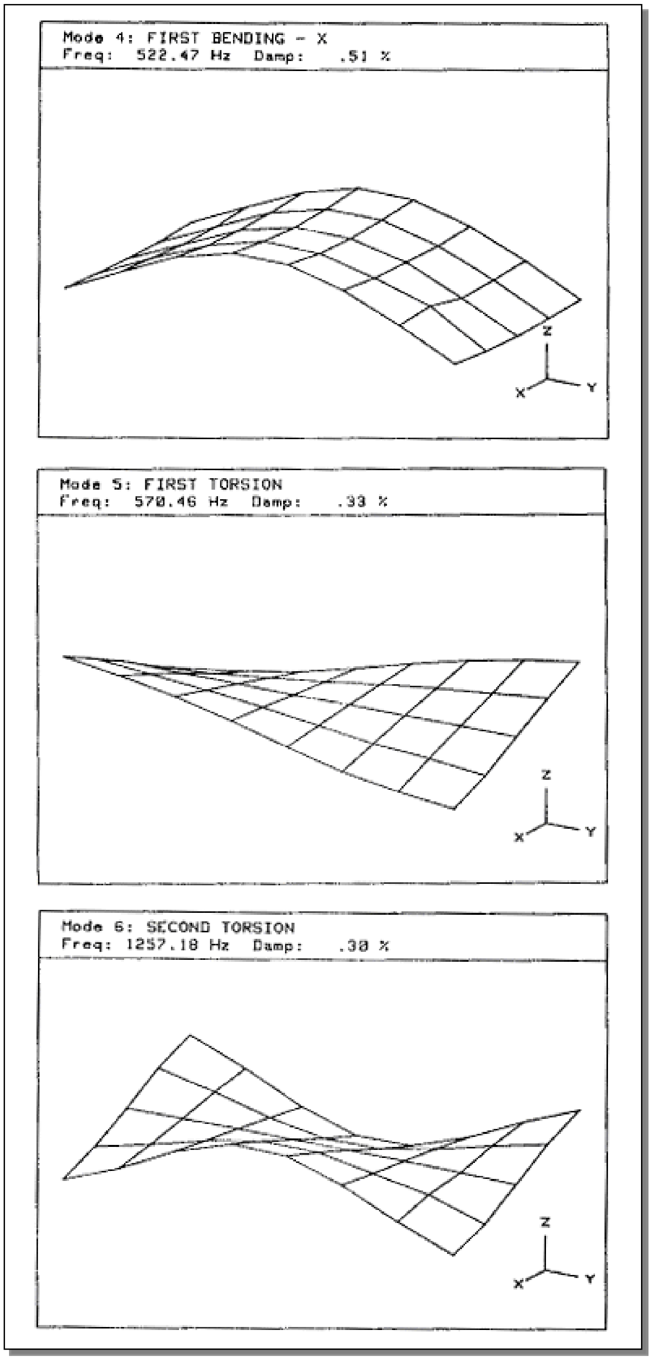
Modes of the Unmodified Structure

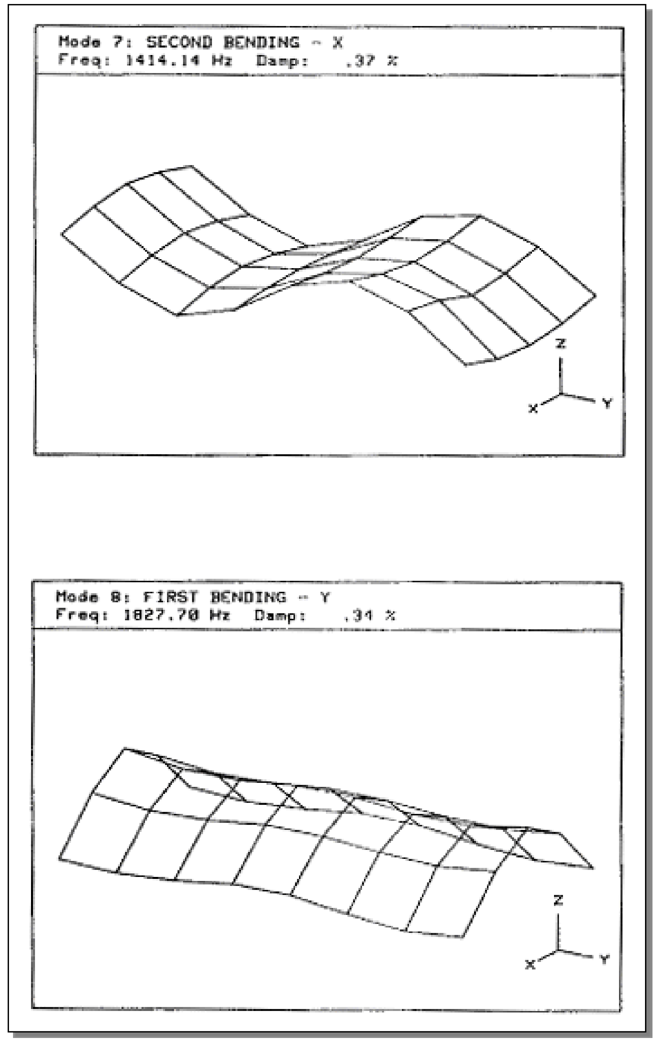
Mode No.	Frequency (Hz)	Damping (%)	Description
1	0.00	0.00	Rigid Z Translation
2	0.00	0.00	Rigid X Rotation
3	0.00	0.00	Rigid Y Rotation
4	522.47	0.51	First Bending - X
5	570.46	0.33	First Torsion
6	1257.18	0.30	Second Torsion
7	1414.14	0.37	Second Bending - X
8	1827.70	0.34	First Bending - Y
9	2156.90	0.27	2^{nd} Order Bending

S^2DM and SDM were first compared by simulating the mounting of the flat plate with springs of stiffness=1000 LBF/inch between its four corners and ground. In this case, SDM solved four scalar eigenvalue problems in sequence (one for each spring addition), while S^2DM found the answer in one eigensolution. The results are shown below.

Modes of the Mounted Structure

	SDM		S^2DM	
Mode No.	Freq (Hz)	Damp (%)	Freq (Hz)	Damp (%)
1	56.49	0.00	55.99	0 00
2	92.36	0.00	92.82	0.00
3	96.02	0.00	97.12	0.00
4	533.96	0.50	533.81	0.50
5	582.35	0.32	582.18	0.32
6	1267.41	0.30	1276.46	0.30
7	1414.37	0.36	1414.37	0.37
8	1831.38	0.34	1831.36	0.34
9	2160.45	0.27	2160.44	0.27



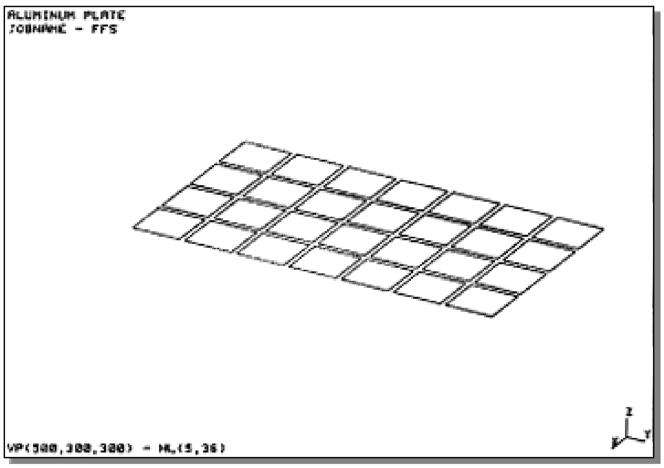


Modes of the Unmodified Structure.

Comparison of Test and Finite Element Analysis

Next, a finite element model of the flat plate was built to compare an eigensolution in physical space with that found by S²DM in modal space.

The plate was modeled using FEDESK, a finite element code which runs on the 9000 Series Hewlett Packard desktop computers. A plot of the finite element model is show below.



First, the free-free flexible body modes were found from the finite element model. The frequencies of these modes are compared with those of the test modes in the table below.

Elastic Modes of the Unmodified Structure

	Test		FEM	
Mode No.	Freq (Hz)	Damp (%)	Freq (Hz)	Damp (%)
1	522.47	0.51	519.50	0.00
2	570.46	0.33	592.16	0.00
3	1257.18	0.30	1278.29	0.00
4	1414.14	0.37	1358.95	0.00
5	1827.70	0.34	1663.98	0.00
6	2156.90	0.27	2001.62	0.00

To compare S²DM and FEDESK, the elastic FEM modes, together with the three rigid body modes, were used as input to S²DM to simulate the mounting of the plate to ground with four springs at its corners. Then, this new set of boundary conditions, (mounting on the plate on four springs), was added to the finite element model, and a new eigensolution was generated using FEDESK. The two sets of results are shown below.

Modes of the Mounted Structure

	S²DM		FEM	
Mode No.	Freq (Hz)	Damp (%)	Freq (Hz)	Damp (%)
1	56.14	0.00	55.99	0.00
2	92.89	0.00	92.74	0.00
3	96.86	0.00	96.73	0.00
4	529.33	0.00	529.27	0.00
5	609.12	0.00	608.99	0.00
6	1287.80	0.00	1287.76	0.00
7	1360.38	0.00	1360.38	0.00
8	1667.22	0.00	1667.20	0.00
9	2008.50	0.00	2008.48	0.00

CONCLUSIONS

The purpose of the paper was to point out the advantages of implementing the SDM method in a different manner than has been done in the past. The SDM method is attractive as an analysis tool because it only requires modal data, which can be derived either from analysis or test, to describe the dynamics of the structure, and it is computationally very efficient. The method proposed here (S²DM) is not as efficient as the original method if only one modification is done, but its comparative efficiency increases rapidly when a large number of scalar modifications must be modeled on a structure.

In the simple example in this paper, only four stiffness modifications were performed in the mounting of a flat plate structure to ground, but even in this case S²DM found the solution more rapidly than either SDM or the finite element method. All three methods were run on the same computer with the following results.

Method	Time to Solve 4 Spring Mounting Problem
SDM	43 seconds
S²DM	32 seconds
FEM	7 minutes, 45 seconds

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