Viscous vs. Structural Damping in Modal Analysis

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Abstract

It is shown that among the various damping mechanisms that are generally encountered in a mechanical structure, only the "viscous component" actually accounts for energy loss. The remaining portion of the damping force is due to non-linearities, which do not dissipate energy. Consequently, a linear model of a vibratory mechanical system involves only viscous damping, although the damping factor may depend upon the waveform or amplitude of the excitation signal.

Introduction

The motion of an elastic mechanical system is commonly modeled in the time domain by the equation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \qquad (1)$$

where **f(t)** is the driving force vector, and **x(t)** is the resulting displacement vector of a system with mass. Damping, and stiffness matrices denoted by **M**, **C**, and **K**, respectively. The dots indicate derivatives with respect to the time variable **(t)**. If we pre-multiply this equation by the transposed velocity vector $\mathbf{v}^t(t) = \dot{\mathbf{x}}^t(t)$, we obtain an instantaneous power balance equation.

$$\mathbf{v}^t(t)\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{v}^t(t)\mathbf{C}\mathbf{v}(t) + \mathbf{v}^t(t)\mathbf{K}\mathbf{x}(t) = \mathbf{v}^t(t)\mathbf{f}(t) \qquad (2)$$

We can integrate this equation over any time interval (τ) of interest to obtain an energy balance equation for the particular time interval. The energy associated with the mass and stiffness matrices is stored energy that can always be recovered, but the portion given by $\int_\tau \mathbf{v}^t(t)\ \mathbf{C}\mathbf{v}(t)\ \mathbf{dt}$, is dissipated, usually in the form of heat, and is lost from the system. In this mathematical formulation, the force $\mathbf{C}\mathbf{v}(t)$ is called a viscous damping force, since it is proportional to velocity. However, as we discuss next, this does not necessarily imply that the physical damping mechanism is viscous in nature. It is important to recognize that the physical damping mechanism and the mathematical model of the mechanism are two distinctly different concepts. The term "viscous" is commonly used indiscriminately to denote both a damping mechanism (i.e. fluid flow), and a mathematical representation of dissipated energy described by a force (i.e. $\mathbf{C}\mathbf{v}(t)$) that is proportional to velocity.

In practice most mechanical structures exhibit rather complicated damping mechanisms, but we will show that all of these can be mathematically modeled by a force proportional to velocity, so that the mathematical usage of the term "viscous" is generally implied**.**

We will show that there is always a viscous component of damping force (proportional to velocity), and that this viscous component accounts for all energy loss from the system. We will see that all remaining force terms are due to non-linearities, and do not cause energy dissipation. Thus, we only need to measure the viscous term to characterize the system using a linear model.

In appendix A, we discuss the current technique of using an imaginary stiffness to model structural damping (for sinusoidal excitation), and in appendix B. we discuss the concept of hereditary damping introduced by Klosterman [4]. Reference [5] is also recommended for a general discussion of damping mechanisms and mathematical models.

Damping Mechanisms

Three of the most common damping mechanisms are: 1.) coulomb (sliding frictions [2] in which the force magnitude is independent of velocity.) 2.) viscous, where force is proportional to velocity, and 3.) structural (hysteretic, internal, material) [3], in which the force is proportional to the magnitude of the displacement from some quiescent position.

From a microscopic point of view, most damping mechanisms involve frictional forces that oppose the motion (velocity) of some part of a physical system, resulting in heat loss**.** For example coulomb friction force is caused by two surfaces sliding with respect to one another, and this sliding force is independent of velocity, once the initial static friction (*stiction*) is overcome.

Structural damping may be viewed as a sliding friction mechanism between molecular layers in a material, in which the friction force is proportional to the deformation or displacement from some quiescent or rest position**. I**magine a rod made of a bundle of axial fibers. The sliding friction

force between each fiber and its neighbor will increase as the rod is bent and the fibers are pinched together. This pinching phenomenon occurs in most materials as the various molecular layers slide past one another. The result is a damping force that is proportional to the displacement from the undisturbed position. This mechanism was verified for a wide range of materials by Kimball and Lovell in 1927 [3].

Viscous damping occurs when molecules of a viscous fluid rub together, causing a resistive friction force that is proportional to, and opposing the velocity of an object moving through the fluid.

We can conveniently characterize damping mechanisms of these types by the force equation.

$$\mathbf{f(t) = -c\,|\,g(t)\,|\,sgn(v)} \qquad (3)$$

Where $\mathbf{f(t)}$ is the damping force, **c** is a scalar damping coefficient, $\mathbf{|g(t)|}$ is some arbitrary magnitude function, and **sgn(v)** is the ***signum*** of velocity defined by

$$\begin{aligned} \mathbf{sgn(v)} &= \mathbf{1}, \quad \text{for } v > 0 \\ &= \mathbf{0}, \quad \text{for } v = 0 \\ &= \mathbf{-1}, \quad \text{for } v < 0 \end{aligned} \qquad (4)$$

We can catalog the three most common damping mechanisms by choosing the appropriate $\mathbf{g(t)}$ as follows:

1) Coulomb: $\mathbf{g = 1,\ f_c(t) = -c\ \ sgn(v)}$ (5)

2) Viscous: $\mathbf{g = v,\ f_v(t) = -c\ v\,|\,sgn(v) = -cv}$ (6)

3) Structural: $\mathbf{g = x,\ f_s(t) = -c\,|x|\,sgn(v)}$ (7)

Notice that only the viscous damping force is a linear function of velocity, and that the other mechanisms are inherently non-linear in nature.

As an example, let's assume that the displacement is sinusoidal so.

$$\mathbf{x(t) = \sin \omega t,} \text{ and} \qquad (8)$$

$$\mathbf{v(t) = \dot{x}(t) = \omega \cos\ \omega t} \qquad (9)$$

The resulting coulomb damping force is obviously a square wave of period $\frac{2\pi}{\omega}$, with peak amplitude **c**. The viscous damping force is a cosine function, and the structural damping force is the product of the coulomb square wave force times $\mathbf{|\sin\ \omega t|}$.

Hysteresis Curves

It is instructive to plot damping force vs. displacement for the three cases mentioned above. These plots are called "hysteresis" curves, and we will see that the areas enclosed by these curves represent the system energy loss per cycle of excitation.

For viscous damping, we obtain the ellipse shown in Figure 1, having an area of $\mathbf{\pi\omega c}$, units of energy loss per cycle.

Coulomb damping yields a rectangular hysteresis curve, shown in Figure 2.

Structural damping causes the "bow tie" curve shown in Figure 3.

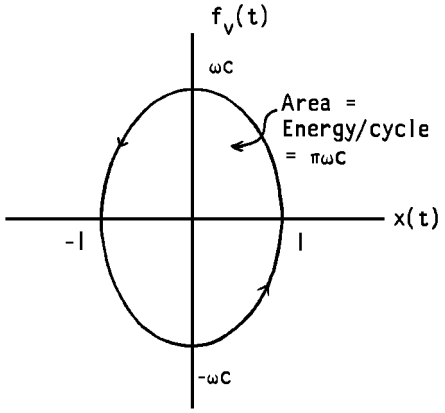

Figure 1. Hysteresis Curve for Viscous Damping

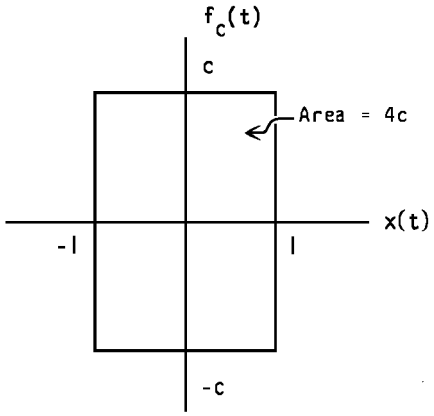

Figure 2. Hysteresis Curve for Coulomb Damping

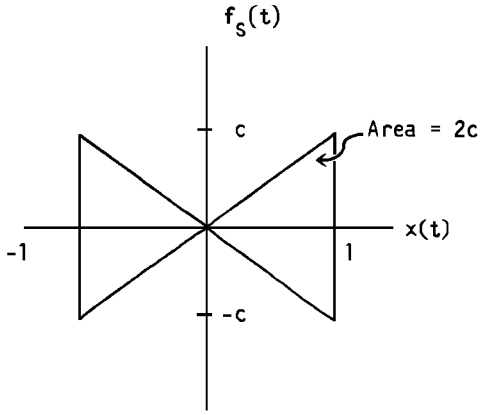

Figure 3. Hysteresis Curve for Structural Damping

An "equivalent viscous" representation can always be obtained by constructing an elliptical hysteresis curve having the same area and corresponding to the same displacement as the actual hysteresis curve. The resulting force is simply the "equivalent viscous" force for that particular mechanism.

The use of the term "hysteretic" damping is somewhat confusing, since all damping mechanisms involve a hysteresis curve of some sort. Thus, we prefer to use the word "structural" to describe this particular mechanism. It should be emphasized that the hysteresis curves, which have been illustrated, are for the case or sinusoidal displacement. The curves will be different if other displacements are used.

By definition, the total energy loss over each cycle of the displacement is given by

$$\Delta\varepsilon = \int_0^{\frac{2\pi}{\omega}} \mathbf{f(t)}\ \mathbf{v(t)}\ \mathbf{dt} = 2\int_{-1}^{1} \mathbf{f(t)}\ \mathbf{dx} \qquad (10)$$

which is simply the area enclosed by the appropriate hysteresis curve. Notice that his area is proportional to frequency for the viscous case, but independent of frequency for the other two mechanisms. This explains why the damping factor of structures having primarily non-viscous damping remains low. Even at high resonant frequencies. If all damping were viscous, then small, high frequency bells would react to a strike with a dull thud, instead of a clear tinkle.

Fourier Series Analysis

Each of these damping forces can be represented by a Fourier series. For the coulomb case, we have

$$\mathbf{F_c}\left(\frac{\mathbf{k\omega}}{\mathbf{2\pi}}\right) = -\ \frac{\mathbf{4c}}{\boldsymbol{\pi}\mathbf{k}}(\mathbf{-1})^{\frac{\mathbf{k-1}}{\mathbf{2}}},\ \mathbf{k = 1,3,5,..} \qquad (11)$$

where ***k*** is the harmonic number, and $\mathbf{F_c}\left(\frac{\mathbf{k\omega}}{\mathbf{2\pi}}\right)$ is the frequency spectrum amplitude of $\mathbf{f_c(t)}$. For structural damping, we can write

$$\mathbf{F_s}\left(\frac{\mathbf{k\omega}}{\mathbf{2\pi}}\right) = -\ \frac{\mathbf{2c}}{\boldsymbol{\pi}}\ \frac{(\mathbf{-1})^{\frac{\mathbf{k-1}}{\mathbf{2}}}}{\mathbf{2}\left[\frac{\mathbf{k}}{\mathbf{4}}\right]+\mathbf{1}},\ \mathbf{k\ = 1,3,5} \qquad (12)$$

where $\left[\frac{\mathbf{k}}{\mathbf{4}}\right]$ is the integer part of $\frac{\mathbf{k}}{\mathbf{4}}$. In both cases, we have odd harmonic cosine terms**.** Only the fundamental term **(k=l**) can dissipate energy, since there is no motion at any of the harmonic frequencies. Thus, the viscous component of each of these forces is defined as simply the fundamental term, and all energy loss is accounted for by this term. This result is derived in a more general way in the next section.

The Equivalent Viscous Component

Let's define **f(t)** to be the damping force vector resulting from some displacement vector **x(t)**, or corresponding velocity vector $\mathbf{v(t)} = \dot{\mathbf{x}}\mathbf{(t)}$. We will not restrict the excitation waveform, so these time functions will be completely general. We will also not restrict our attention to any particular damping mechanism, but will allow any linear or non-linear damping force vector that may occur. The energy loss over any arbitrary time interval $\boldsymbol{\tau}$ is defined to be

$$\mathbf{E} = \int_{\tau} \mathbf{f}^{\mathbf{t}}\mathbf{(t)v(t)\ dt} \qquad (13)$$

where the superscript **t** denotes the transpose of the vector. If the excitation is periodic, than $\boldsymbol{\tau}$ will generally be chosen as some multiple of the period. For random excitation, we would choose $\boldsymbol{\tau}$ to be long compared to the width of the autocorrelation function of the exciting signal.

Let's define a viscous force vector $\mathbf{f_v(t)}$ to be proportional to velocity, as follows

$$\mathbf{f_v(t)\ =\ b\ v(t)} \qquad (14)$$

where **b** is some scalar proportionality constant to be determined by a least-squares fit of $\mathbf{f_v(t)}$ to $\mathbf{f(t)}$, as we describe next. Define the squared error ***c*** between the observed damping force and the desired equivalent viscous force component by

$$\mathbf{c} = \int_{\tau} (\mathbf{f} - \mathbf{bv})^{t} \ (\mathbf{f} - \mathbf{bv})\, \mathbf{dt} \qquad (15)$$

Next, we choose **b** to minimize **c** by setting $\frac{\mathbf{dc}}{\mathbf{db}}$ to zero, obtaining

$$\frac{\mathbf{dc}}{\mathbf{db}} = \mathbf{0} = -2 \int_{\tau} (\mathbf{f} - \mathbf{bv})^{t}\, \mathbf{dt} \qquad (16)$$

from which we get

$$\mathbf{b} = \frac{\int_{\tau} \mathbf{f}^{t} \mathbf{v} \mathbf{dt}}{\int_{\tau} \mathbf{v}^{t} \mathbf{v} \mathbf{dt}} = \frac{\mathbf{E}}{\int_{\tau} \mathbf{v}^{t} \mathbf{v} \mathbf{dt}} \qquad (17)$$

where we have used equation (13) for **E**. Thus, the least-squares equivalent viscous force is

$$\mathbf{f_v} = \mathbf{bv} = \frac{\mathbf{Ev}}{\int_{\tau} \mathbf{v}^{t} \mathbf{v} \mathbf{dt}} \qquad (18)$$

Now, the energy loss due to $\mathbf{f_v}$ is given by

$$\mathbf{E_v} = \int_{\tau} \mathbf{f_v}^{t} \mathbf{v} \mathbf{dt} = \frac{\mathbf{E} \int_{\tau} \mathbf{v}^{t} \mathbf{v} \mathbf{dt}}{\int_{\tau} \mathbf{v}^{t} \mathbf{v} \mathbf{dt}} = \mathbf{E} \qquad (19)$$

We see that the energy dissipated by the viscous force $\mathbf{f_v(t)}$ in the interval τ exactly the same as the original energy dissipated by the genera1 damping force $\mathbf{f(t)}$. Therefore, the remaining non-viscous force ($\mathbf{f} - \mathbf{f_v}$) does not dissipate any energy, and is said to be orthogonal to $\mathbf{v(t)}$ over the interval τ. This is expressed as

$$\int_{\tau} (\mathbf{f} - \mathbf{f_v})^{t} \mathbf{v} \mathbf{dt} = \mathbf{0} \qquad (20)$$

For coulomb and structural damping mechanisms, this non-viscous force represents non-linear system behavior, and contains harmonics and inter-modulation products of the input excitation that have no velocity counter parts, and hence cannot dissipate energy. We should emphasize that the value of **b** generally depends upon the particular waveforms of **v(t)** and **f(t)** and hence varies with excitation level, and with the type of exciting signal. For example, the effective viscous damping factor may be different for random excitation than for sinusoidal excitation, even though the same **rms** (root-mean-square) signal level is used in both cases. This situation is common for all non-linear systems, and implies that any linear approximation is only valid for the particular conditions under which the approximation was originally determined.

Other damping mechanisms besides coulomb and structural damping undoubtedly exist, particularly for high velocity airfoils, or if hydraulic or iron core magnetic sub-systems are included in the system being modeled. The previous result applies for any damping mechanism, and could also be used to linearly model stiffness or mass.

Implications for Modal Analysis

The object of modal analysis [1] is to obtain the natural frequency, damping factor, and geometrical mode shape of each of several modes of vibration, until we have adequately characterized the system behavior over some frequency band of interest. We assume a linear system for this modal representation, and if the system is somewhat non-linear, we generally want the best linear approximation to the actual system that we can obtain. Each mode can then be modeled at each point on a structure as a single degree-of-freedom resonator, having some natural frequency, and some damping factor, which are common to all points on the structure.

In terms of the Laplace transform, we represent the $\mathbf{k^{th}}$ complex mode of vibration by a function of the form

$$\mathbf{H_k(s)} = \frac{\mathbf{A_k}}{\mathbf{s} - \mathbf{p_k}} + \frac{\mathbf{A_k^*}}{\mathbf{s} - \mathbf{p_k^*}} \qquad (21)$$

where $\mathbf{p_k}$ is the "pole" (or singularity) location in the **s**-plane, and $\mathbf{A_k}$ is called the "residue" of the pole. Natural frequency and the damping coefficient are obtained from the imaginary part and real part respectively, of $\mathbf{p_k}$. The resulting impulse response in the time domain is

$$\mathbf{h_k(t)} = \mathbf{A_k e^{p_k t}} + \mathbf{A_k^* e^{p_k^* t}} \qquad (22)$$

Each complex pole always has a conjugate mate, as indicated by the two terms in equations (21) and (22).

When we measure the modal parameters of some physical structure, we obtain "best" estimates of $\mathbf{A_k}$ and $\mathbf{p_k}$ (perhaps using a least-squares estimation procedure) so that the effects of non-linearities and noise are reduced as much as possible. Thus, no matter what damping mechanism is

involved, we will obtain a best linear estimate of the pole location $\mathbf{p_k}$, which will imply some equivalent viscous damping coefficient.

However, we have seen that this viscous damping estimate will properly account for all energy loss in the system, so it doesn't matter what the actual damping mechanism is, as far as a linear representation of the system is concerned. This is not to say that non-linearities are unimportant, but their study is generally very difficult, and beyond the scope of present day practice in modal analysis.

Summary and Conclusions

Three commonly occurring damping mechanisms: (1) coulomb, (2) viscous, and (3) structural damping were identified and classified. We have seen that a coulomb damping force is independent of excitation magnitude, so a system with primarily coulomb damping will show a damping factor that decreases with stronger excitation. On the other hand, both viscous and structural damping forces are proportional to excitation so their associated damping factors will remain constant with excitation level. Indeed, it is nearly impossible to distinguish between viscous and structural damping mechanisms, without physically altering the structure. If the structure can be altered to raise the resonant frequencies, then the viscous damping factors will increase, while the structural damping factors will remain unchanged.

In a very qualitative way, the regions that are dominated by each particular damping mechanism are as follows:

	Low Frequencies	High Frequencies
Small Excitation	Coulomb	Viscous or Coulomb
Large Excitation	Structural	Viscous

Figure 4. Dominant Regions for each Damping Mechanism

We concluded that coulomb and structural mechanisms imply non-linear system behavior, but that an equivalent viscous damping component can always be derived, which will account for all of the energy loss from the system. Thus, in measuring the modal vibration parameters for the linear motion of a system, we don't care what the detailed damping mechanism really is.

It is shown in the two appendices that hereditary damping is a way of representing the composite energy loss of several closely coupled modes, and that the current practice of representing structural damping by introducing an imaginary stiffness is either not physically realizable, or else introduces new fictitious modes, but in either case, actually involves only viscous damping.

APPENDIX A

The Use of Imaginary Stiffness to Represent Structural Damping.

Since structural damping force is proportional to displacement and independent of frequency, it seems logical to model this damping mechanism as a stiffness term in the system equations. It must however be imaginary to provide a 90-degree phase angle relative to the ordinary stiffness contribution, and hence have the appearance of a damping mechanism.

This formulation results in differential system equations with complex coefficients in the time domain, implying a complex time domain solution**,** and a non-hermitian frequency spectrum. In the Laplace domain, this means that poles do not appear in conjugate pairs, but tend to appear as shown in Figure 5.

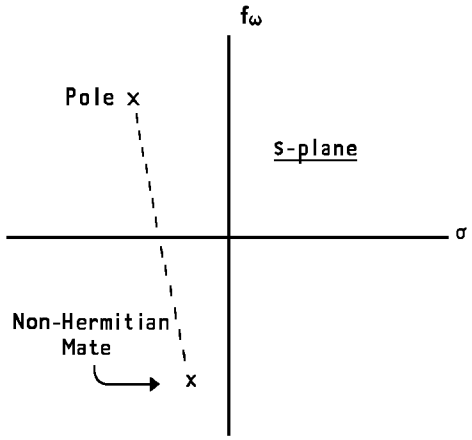

Figure 5. Pole Locations for Non-Hermitian System

It is possible to choose only the hermitian part of the spectrum, thus insuring a real time function, but this is accomplished only by adding an auxiliary pair of poles as shown in Figure 6.

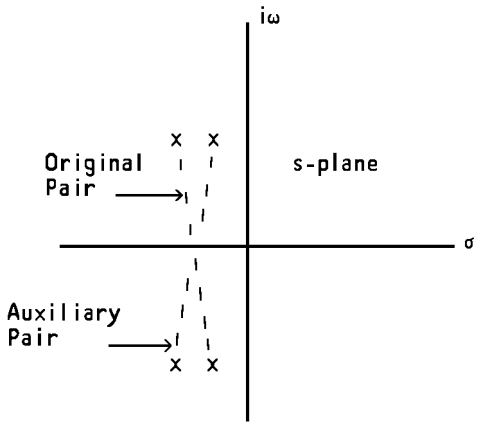

Figure 6. Pole Locations for Hermitian Portion of Non-Hermitian System

We see that this approach results in twice as many poles, and hence twice as many modes as actually exist.

Thus, the use of imaginary stiffness either results in a physically unrealizable system, or else introduces superfluous modes into the system model. Furthermore, notice that we are still working with a linear system, in which a viscous damping coefficient is associated with each pole, so we really have added nothing new in the way of a damping mechanism.

The fact that a viscous damping force is proportional to frequency is relatively unimportant, since the frequency range of interest for each mode of vibration is a narrow band centered around the resonant frequency, which is a constant value. If we want a model in which the damping factor is a constant value for all resonant frequencies, then the viscous model with all poles placed on a common radial line from the origin is a more straightforward approach. Finally, since displacement and velocity are proportional at any (constant) resonant frequency, we see that viscous force is proportional to displacement in any case.

As a consequence of these arguments, there seems to be very little reason to model structural damping via an imaginary stiffness contribution. A much more straightforward approach is to simply model the linear part of structural damping with an equivalent viscous component, as we discuss in the body or this paper.

APPENDIX B
Hereditary Damping

A.L. Klosterman [4] has introduced the concept of hereditary damping, in an attempt to account for damping mechanisms that are more general than simple viscous damping. Klosterman defined a heredity function $\phi(t)$, and postulated that the damping force could be written as

$$f_h(t) = -c\,\phi(t) * v(t) \qquad (23)$$

where the asterisk denotes the time domain convolution between $\phi(t)$ and the velocity waveform.

In the Laplace domain, we can write the damping force as

$$F_h(s) = -c\,\phi(s) \quad V(s) \qquad (24)$$

where F_h, ϕ, and **V** are the respective Laplace transforms of f_h, ϕ, and **v**. Obviously, viscous damping is obtained for the special case when $\phi(s) = 1$, or $\phi(t) = \delta(t)$, where $\delta(t)$ is the unit delta function.

In a sense, it appears that this represents a generalization of various linear damping mechanisms, but we will see that it is actually a way of describing the composite damping force due to several poles clustered together in some region of the **s**-plane. Each pole in the cluster displays ordinary viscous damping, so no new damping mechanisms are defined by this concept.

If we take the Laplace transform of equation (1), replacing the viscous force with the hereditary force, we get

$$\left(\mathbf{Ms^2 + Cs\,\phi(s) + K}\right) \quad \mathbf{X(s) = F(s)} \qquad (25)$$

where **X** and **F** are the respective Laplace transforms of **x(t)** and **f(t)**. In this equation, **M**, **C**, **K** and ϕ are matrices, and **F** and **X** are vectors. The pole locations are given by the values of **s** for which

$$\mathbf{Det}\left[\mathbf{Ms^2 + Cs\,\phi(s) + K}\right] = \mathbf{0} \qquad (26)$$

Let's assume that elements of $\phi(s)$ are rational fractions in **s**. Then, we see that the total number of poles in the linear system can be greatly multiplied depending on the numerator and denominator of ϕ. Now, since **Cs** is generally small compared to $\mathbf{Ms^2}$ and **K**. the basic pole locations remain relatively unchanged, no matter that form $\phi(s)$ takes. For each pole corresponding to $\phi = 1$, we have a cluster of poles in the same general vicinity for some $\phi = (s) \neq 1$, since only small perturbations on the original pole values are needed $\left(\mathbf{via\ Ms^2}\right)$ to counteract the new contributions from $\mathbf{Cs\,\phi(s)}$. Thus, although hereditary

damping may be a useful concept, it does not represent any new damping mechanism and can be eliminated, at least in principle, by simply measuring the individual pole locations, along with the modal vectors associated with each pole.

References:

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