OBTAINING STRESSES AND STRAINS FROM ODS DATA

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ABSTRACT

During an Operating Deflection Shape (ODS) test, there is often concern not only about the displacements that occur during operation, but also about the stress or strain levels that are being encountered. If in-plane vibration were measured, then strain could be estimated as the change in displacement between two transducers divided by the distance between them. Unfortunately, vibration is usually measured normal to the surface instead of in-plane, so stress or strain cannot be calculated from this data.

However, if a finite element model is used in conjunction with ODS test data, the model can be deformed using the measured data and the appropriate stresses, strains and even applied forces can be calculated. Details of this method and several practical examples of its use are included in this paper.

OBTAINING STRAINS FROM ODS's

When performing an ODS test, we can measure ODS FRFs (magnitudes & phases) at several DOFs and obtain a set of displacements that describe the deformation of a structure, at any frequency [6]. The question then becomes, "For a given ODS of structural deformations, what are the associated stress and strain levels being experienced by the structure?"

Since strain is defined as the rate of change in the deformation, it can always be calculated from displacements. For the simple case of a rod being extended and compressed along its length, its strain would be calculated with the formula,

$$\varepsilon = \frac{x}{L} \qquad (1)$$

where:

x = the amount of deflection of the rod.
L = the length of the rod.

Stress is simply related to strain by the modulus of elasticity,

$$\sigma = E\varepsilon \qquad (2)$$

In most practical cases however, the strain field is more complex and includes the effects of structural bending and torsion. For these cases, the equations relating displacement and strain are much more complex, but finite element analysis can be used to obtain a solution.

Finite element analysis is commonly used to determine stress or strain levels due to certain static loading conditions on a structure. Using this same approach, a finite element model can also be used with experimental ODS data to calculate the stresses and strains being experienced within a structure.

BACKGROUND THEORY

The simplest model of a vibrating structure is a single-degree-of-freedom (SDOF) system consisting of a mass connected by a spring and damper to ground.

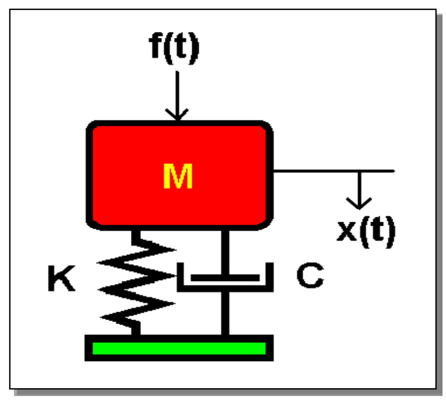

Figure 1. Mass-Spring-Damper (SDOF) Structure.

The motion of this structure is governed by Newton's Second Law,

$$f(t) = M\ddot{x}(t) + C\dot{x}(t) + Kx(t) \qquad (3)$$

where:

M = the distributed mass of the structure.
C = the damping within the structure.
K = the stiffness of the structure.

f(t) = the force applied to the structure over time.
x(t) = the displacement of the structure over time.
$\dot{x}(t)$ = the velocity of the structure over time.
$\ddot{x}(t)$ = the acceleration of the structure over time.

In the frequency domain, the equivalent equation of motion is,

$$\mathbf{F(s) = [M\, s^2 + C\, s + K]\ X(s)} \tag{4}$$

where:

$\mathbf{X(s)}$ = Laplace transform of the displacement.
$\mathbf{F(s)}$ = Laplace transform of the force.
$\mathbf{s = \sigma + j\omega =}$ complex Laplace variable

Assuming a quasi-static solution and removing the time variable from equation (3), a simplified form of Newton's Second Law relates the amount of deformation directly to the applied forces,

$$\mathbf{f = K\, x} \tag{5}$$

This is also known as Hooke's Law. Since the force and displacement are assumed to be constant (quasi-static) with respect to time, the mass and damping terms (involving in the velocity and acceleration) are negligible and are therefore not required.

Complex Structures

Newton's Second Law still governs the motion of more complex structures, but instead of a single equation like (3) or (4), multiple equations are written. The mass, stiffness, and damping are replaced with matrices of constants, and the force and motion terms become vectors instead of scalars.

Solid Mechanics has been used to derive closed form solutions to equations (3) or (4) for a variety of simplified geometries. Formulas to obtain stresses and strains in beams and plates are compiled in Reference [1]. Formulas to obtain the dynamic characteristics of beams and plates are compiled in Reference [2].

In general, it is impossible to derive a closed form solution for a complex structure such as a machine or a bridge. However, finite element modeling can be used to sub-divide a complex structure into a series of simple elements such as beams or plates. The stiffnesses of these simple elements can be represented using known analytical formulae. Many interconnected finite elements can then be used to form a set of matrix equations that define the dynamics of complex structures.

Each finite element provides a description of how the structure behaves in its local region. The stiffness matrix for the overall structure is then constructed by calculating element stiffnesses and also taking into account how each of the elements interacts with its surrounding neighbors (i.e. its boundary conditions).

A typical finite element is defined by points and straight line boundaries connecting the points. A set of equations can then be defined for the displacement anywhere inside the boundaries of the element, based on the displacements of its endpoints. Since displacements can be defined anywhere in an element, stresses and strains can also be defined anywhere. The general equation for the stiffness matrix of an element is,

$$\mathbf{[K] = \int_V [B]^T [D][B] dV} \tag{6}$$

where:

$\mathbf{[D]}$ = elasticity matrix.
$\mathbf{[B]}$ = strain matrix.

The strain matrix $\mathbf{[B]}$ transforms the displacements into strains and the elasticity matrix $\mathbf{[D]}$ transforms the strains into stresses. The form of the strain matrix and the elasticity matrix will vary based upon the type of finite element used. The elasticity matrix will also vary based upon the material model being used (Isotropic, Orthotropic, etc.). Strain matrices for different element types as well as numerical integration techniques can be found in most finite element text books [3], [4].

The finite element method generates stiffness matrices for each element, and then sums them together to create a global stiffness matrix. For complex structures, equation (5) becomes,

$$\begin{aligned}\mathbf{F_{global}} &= \mathbf{[K]_{global}\, x_{global}} \\ &= \mathbf{\left(\sum [K]_{local}\right) x_{global}}\end{aligned} \tag{7}$$

To solve equation (7), it is assumed that either the displacement of a degree-of-freedom (direction at a point) is known, or the applied force at the DOF is known. This allows the solution of equation (7) to be partitioned as follows,

$$\begin{Bmatrix}\mathbf{f_u} \\ \mathbf{f_k}\end{Bmatrix} = \begin{bmatrix}\mathbf{K_{11}} & \mathbf{K_{12}} \\ \mathbf{K_{21}} & \mathbf{K_{22}}\end{bmatrix}\begin{Bmatrix}\mathbf{x_k} \\ \mathbf{x_u}\end{Bmatrix} \tag{8}$$

where:

$\mathbf{f_u}$ = unknown forces.

$\mathbf{f_k}$ = known forces.

$\mathbf{x_u}$ = unknown displacements.

$\mathbf{x_k}$ = known displacements.

Equation (8) can be solved for the unknown displacements as follows,

$$\begin{aligned}&\mathbf{K_{21}x_k + K_{22}x_u = f_k} \\ &\mathbf{K_{22}x_u = f_k - K_{21}x_k} \\ &\mathbf{x_u = [K_{22}]^{-1}\{f_k - K_{21}x_k\}}\end{aligned} \tag{9}$$

Equations (9) shows that the unknown displacements $\mathbf{x_u}$ can be calculated from the known forces $\mathbf{f_k}$ and known displacements $\mathbf{x_k}$. Once all of the displacements in the structure are known, equation (8) can be solved to determine the unknown applied forces $\mathbf{f_u}$. Or we can go back to the element matrices in (6) and compute stresses and strains within the elements.

SHAPE EXPANSION USING FINITE ELEMENTS

From equation (9) it is clear that the deformations at unknown DOFs $\mathbf{x_u}$ are based not only on any known applied forces, but also on known (measured) deformations.

In general, since only displacements are measured in an ODS test, all of the known forces are assumed to be zero. This assumption further simplifies equation (9) to,

$$\mathbf{x_u} = -[\mathbf{K_{22}}]^{-1}[\mathbf{K_{21}}]\{\mathbf{x_k}\} \quad (10)$$

Equation (10) can therefore be used to extend (interpolate or extrapolate) shape data to all unmeasured DOFs using data from the measured DOFs.

Illustrative Example No. 1

Consider a **25 in. long 1 in. square** steel cantilever beam deflected **2 in.** at its free end, as shown in Figure 2. Applying the 2 in. deflection as a prescribed (measured) displacement and solving this problem using the NASTRAN finite element program, shows that a **320 lb.** force must be applied at the free end of the beam to deflect it 2 in.

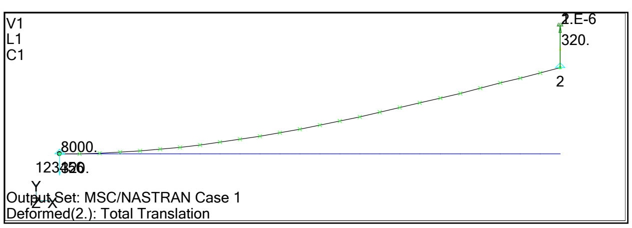

Figure 2. Cantilever Beam Deflected by an End Load.

The moments in the beam start at zero at the free end, and increase linearly to **8000 in-lbs** at the fixed end.

The maximum bending stress at the fixed end of beam is,

$$\sigma = \frac{\mathbf{Mc}}{\mathbf{I}} = \frac{\mathbf{8000 * 0.5}}{\mathbf{(1)^4/12}} = \mathbf{48000\ psi}$$

where:

M = moment of inertia.
c = distance from neutral axis to the top or bottom surface.
I = cross sectional inertia.

With 2 in. of prescribed displacement at the free end, the displacement at the midpoint of the beam is calculated as **0.625 in**.

If this displacement is applied to the finite element model instead of the 2 in. displacement at the free end, a different solution will result, as shown in Figure 3.

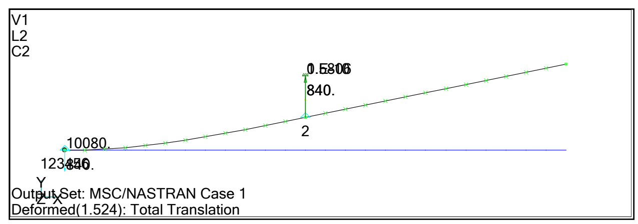

Figure 3. Cantilever Beam Deflected by a Center Load.

Moving the prescribed displacement also moves the location of the applied force. Instead of applying a force at the free end, it would be applied at the same location as the prescribed displacement, at the midpoint.

Notice that a 0.625 in. prescribed displacement at the midpoint displaces the free end only **1.524 in.** instead of 2 in. Since the load is applied at the midpoint and the beam is not subjected to any other forces, it therefore remains straight to the right of the midpoint.

When the beam is displaced from its midpoint, the forces, moments and stresses in the beam will clearly be different than when the free end is displaced. To displace the midpoint by 0.625 in., an **800 lb.** force must be applied at the midpoint. The moment at the fixed end of the beam increases to **10000 in-lbs** and the peak bending stress increases to **60000 psi.**

This example shows that a finite element solution is possible if displacements are measured at all of the DOFs where forces are applied to the structure. Otherwise, the load will be redistributed and the calculated deformations, stresses and strains will be incorrect.

Nevertheless, this method provides improvements over purely geometrical interpolation methods that determine the motions of unmeasured DOFs using the motions of nearby measured DOFs. For the above case, most geometric interpolation methods would yield a straight line deformation between the free and fixed end points. Furthermore, once the motions of the unknown DOFs have been calculated, the finite elements can be used again to calculate stresses and strains.

IMPROVED SHAPE EXPANSION

The standard finite element solution (using the example above) assumes that either the force or the displacement is known at each DOF. However, this does not necessarily need to be true to obtain a solution using equation (8). All that is required is that the *total number of unknown displacements and forces must equal the total number of equations*. This restriction is still satisfied if a measured displacement is applied to one DOF, and a force is specified at a different DOF.

To clarify this, the stiffness matrix can be re-written using 4 separate partitions.

1. ***DOFs with known displacements and unknown forces***. This includes all DOFs that act as boundary conditions, and any DOFs where ODS data is measured, and forces are applied but unknown.
2. ***DOFs with known displacements and known forces***. For most vibration tests, these are the DOFs where ODS data is measured, but where no external forces are applied.
3. ***DOFs with unknown displacements and unknown forces***. These are the DOFs at which displacements are not measured, and forces are applied but also unknown. The number of DOFs of this type will equal the number of DOFs in partition 2.
4. ***DOFs with unknown displacements and known forces***. These are the DOFs for which displacements are not measured, and no forces are applied.

Equation (8) now expands into,

$$\begin{Bmatrix} \mathbf{f_1} \\ \mathbf{0} \\ \mathbf{f_3} \\ \mathbf{0} \end{Bmatrix} = \begin{bmatrix} \mathbf{K_{11}} & \mathbf{K_{12}} & \mathbf{K_{13}} & \mathbf{K_{14}} \\ \mathbf{K_{21}} & \mathbf{K_{22}} & \mathbf{K_{23}} & \mathbf{K_{24}} \\ \mathbf{K_{31}} & \mathbf{K_{32}} & \mathbf{K_{33}} & \mathbf{K_{34}} \\ \mathbf{K_{41}} & \mathbf{K_{42}} & \mathbf{K_{43}} & \mathbf{K_{44}} \end{bmatrix} \begin{Bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \\ \mathbf{x_4} \end{Bmatrix} \qquad (11)$$

Partitioning this matrix and using the known forces and displacements to solve for the unknown displacements gives,

$$\begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} = \begin{bmatrix} \mathbf{K_{21}} & \mathbf{K_{22}} \\ \mathbf{K_{41}} & \mathbf{K_{42}} \end{bmatrix} \begin{Bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{Bmatrix} + \begin{bmatrix} \mathbf{K_{23}} & \mathbf{K_{24}} \\ \mathbf{K_{43}} & \mathbf{K_{44}} \end{bmatrix} \begin{Bmatrix} \mathbf{x_3} \\ \mathbf{x_4} \end{Bmatrix}$$

$$\begin{bmatrix} \mathbf{K_{23}} & \mathbf{K_{24}} \\ \mathbf{K_{43}} & \mathbf{K_{44}} \end{bmatrix} \begin{Bmatrix} \mathbf{x_3} \\ \mathbf{x_4} \end{Bmatrix} = -\begin{bmatrix} \mathbf{K_{21}} & \mathbf{K_{22}} \\ \mathbf{K_{41}} & \mathbf{K_{42}} \end{bmatrix} \begin{Bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{Bmatrix} \qquad (12)$$

$$\begin{Bmatrix} \mathbf{x_3} \\ \mathbf{x_4} \end{Bmatrix} = -\begin{bmatrix} \mathbf{K_{23}} & \mathbf{K_{24}} \\ \mathbf{K_{43}} & \mathbf{K_{44}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{K_{21}} & \mathbf{K_{22}} \\ \mathbf{K_{41}} & \mathbf{K_{42}} \end{bmatrix} \begin{Bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{Bmatrix}$$

In order to solve equation (12), the inverse of a non-symmetric matrix that is the size of the number of unknown DOFs must be calculated. This matrix is also not sparsely populated, so the usual finite element solution methods that apply to symmetric, sparse matrices cannot be used. Consequently, our ability to solve problems with large numbers of unknown DOFs is limited.

It is also possible to create a singular matrix that cannot be inverted. This can be done by clustering the unknown forces together, and isolating the unknown DOFs from the forces.

Illustrative Example No. 2

Consider a stepped aluminum rod subjected to an end load with a measured deflection of **5 mils** at point 4 (the mid-point).

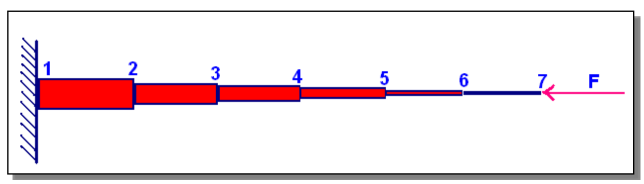

Figure 4. Cantilever Beam with Variable Cross Section.

For a Rod with fixed cross section, the stiffness matrix is,

$$\mathbf{K} = \frac{\mathbf{EA}}{\mathbf{L}} \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix} \qquad (13)$$

where:

L = the Rod length.
A = the Rod cross sectional area.
E = Modulus of elasticity.

The stresses and strains in the Rod are

$$\begin{aligned} \varepsilon &= \frac{\Delta \mathbf{u}}{\mathbf{L}} \\ &= \frac{(\mathbf{u_1} - \mathbf{u_2})}{\mathbf{L}} \qquad (14) \\ \sigma &= \mathbf{E}\varepsilon \end{aligned}$$

Consider the six cross sections of the beam in Figure 4 to have the following properties,

$\mathbf{L}$=10in
$\mathbf{E}$=10^7psi

$$\mathbf{A_{1-2} = 10\ in^2, A_{2-3} = 8\ in^2, A_{3-4} = 6\ in^2}$$
$$\mathbf{A_{4-5} = 4\ in^2, A_{5-6} = 2\ in^2, A_{6-7} = 1\ in^2}$$

These parameters were used to assemble the partitioned stiffness matrix shown in equation (11). Rows 1 & 7 of the stiffness matrix are not used because they correspond to the unknown forces, $\mathbf{F_1}$ & $\mathbf{F_7}$. Columns 1 & 4 are partitioned to be multiplied by the known displacements, $\mathbf{u_1 = 0}$ & $\mathbf{u_4 = 0.005}$. The remaining rows & columns form the partition of the stiffness matrix to be inverted.

$$\begin{Bmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_7 \end{Bmatrix} = \frac{10^7}{10} \begin{bmatrix} 10 & -10 & 0 & 0 & 0 & 0 & 0 \\ -10 & 18 & -8 & 0 & 0 & 0 & 0 \\ 0 & -8 & 14 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & 10 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0.005 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} \quad (15)$$

Solving (15) for the unknown displacements gives,

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 10^6 \left(\begin{bmatrix} -10 & 0 \\ 0 & -6 \\ 0 & 10 \\ 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix} + \begin{bmatrix} 18 & -8 & 0 & 0 & 0 \\ -8 & 14 & 0 & 0 & 0 \\ 0 & -6 & -4 & 0 & 0 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & -2 & 3 & -1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} \right)$$

$$-\begin{bmatrix} -10 & 0 \\ 0 & -6 \\ 0 & 10 \\ 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix} = \begin{bmatrix} 18 & -8 & 0 & 0 & 0 \\ -8 & 14 & 0 & 0 & 0 \\ 0 & -6 & -4 & 0 & 0 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & -2 & 3 & -1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} = -\begin{bmatrix} 18 & -8 & 0 & 0 & 0 \\ -8 & 14 & 0 & 0 & 0 \\ 0 & -6 & -4 & 0 & 0 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & -2 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -10 & 0 \\ 0 & -6 \\ 0 & 10 \\ 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} = \begin{Bmatrix} 1.2766\text{x}10^{-3} \\ 2.8723\text{x}10^{-3} \\ 8.1915\text{x}10^{-3} \\ 0.0145745 \\ 0.0273404 \end{Bmatrix} \quad (16)$$

The forces $\mathbf{F_1}$ & $\mathbf{F_7}$ can now be calculated from the displacements.

$$\begin{aligned} \mathbf{F_1} &= \sum_{i=1}^{7} \mathbf{K_{1i} u_i} \\ &= \mathbf{K_{11} u_1 + K_{12} u_2} \\ &= \frac{10^7}{10}\left(10 * 0 + -10 * 1.2766\text{x}10^{-3}\right) \\ &= -12766 \text{ lb} \\ \mathbf{F_7} &= \sum_{i=1}^{7} \mathbf{K_{7i} u_i} \\ &= \frac{10^7}{10}\left(-1 * 0.0145745 + 1 * 0.0273404\right) \\ &= 12766 \text{ lb} \end{aligned}$$

As expected for a static solution, the applied forces are equal and opposite. This is known as static equilibrium.

Point	u(in)	**Δu** (in)	**ε** (in/in)	**σ** (psi)
1	0.0000			
		0.0012766	1.2766E-4	1277
2	0.0012766			
		0.0015957	1.596E-4	1596
3	0.0028723			
		0.0021277	2.128E-4	2128
4	0.005			
		0.0031915	3.192E-4	3192
5	0.0081915			
		0.006383	6.383E-4	6383
6	0.0145745			
		0.012766	1.277E-3	12766
7	0.0273404			

Table 1. Calculated Results for Cantilever Beam.

For this solution, a 5x5 non -symmetric matrix was inverted. A finite element solution would have partitioned the stiffness matrix as follows and stored the data in banded form allowing for a much more efficient storage of data.

$$\mathbf{K} = \frac{10^7}{10}\left[\begin{array}{c|cc|c|ccc} 10 & -10 & 0 & 0 & 0 & 0 & 0 \\ \hline -10 & 18 & -8 & 0 & 0 & 0 & 0 \\ 0 & -8 & 14 & -6 & 0 & 0 & 0 \\ \hline 0 & 0 & -6 & 10 & -4 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\mathbf{K}_{22} = \frac{10^7}{10}\begin{bmatrix} 18 & -8 & 0 & 0 & 0 \\ -8 & 14 & 0 & 0 & 0 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{K}_{\mathbf{banded}} = \frac{10^7}{10}\begin{bmatrix} 18 & 14 & 6 & 3 & 1 \\ -8 & 0 & -2 & -1 & 0 \end{bmatrix}$$

Storing the data in band form would have reduced the required memory by 60%. Finite element solution methods also employ bandwidth optimizers that allow them to optimize data storage. Memory reductions of 10 to 1 are often achieved. Solutions that require over 1,000MB of computer memory can be solved using less than 100MB of memory.

As previously stated, this method may not always provide a solution. For instance, in Illustrative Example No.2 above, if a force were also been applied to **point 6**, the matrix to be inverted would have a column of zeroes and would be singular. Of course this could be alleviated by specifying the displacement of **point 7**.

The sum of the forces applied at points 6 & 7 could then be calculated, along with the displacements of the other points. However, the displacement of point 7 and the distribution of forces between points 6 & 7 still could not be determined.

USING VELOCITY OR ACCELERATION DATA

Accelerations or velocities are more commonly measured than displacements in an ODS test. This is not a problem however, since it is straightforward to convert from one set of motions to another in the frequency domain.

To convert from displacement to velocity in the frequency domain, the Laplace transform is used for differentiation [5].

$$\begin{aligned} L\{F'(t)\} &= sf(s) - F(0) \\ \dot{X}(s) &= sX(s) \\ \dot{X}(j\omega) &= j\omega X(j\omega) \\ \ddot{X}(j\omega) &= j\omega \dot{X}(j\omega) \\ &= (j\omega)^2 X(j\omega) \\ &= -\omega^2 X(j\omega) \end{aligned} \tag{17}$$

Multiplying the left and right hand sides of equation (17) by $j\omega$ results in the following equations for using velocity measurements,

$$\begin{aligned} j\omega\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix} &= -j\omega\begin{bmatrix} K_{23} & K_{24} \\ K_{34} & K_{44} \end{bmatrix}^{-1}\begin{bmatrix} K_{21} & K_{22} \\ K_{41} & K_{42} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ \begin{Bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} &= -\begin{bmatrix} K_{23} & K_{24} \\ K_{34} & K_{44} \end{bmatrix}^{-1}\begin{bmatrix} K_{21} & K_{22} \\ K_{41} & K_{42} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \end{aligned} \tag{18}$$

For using acceleration measurements,

$$\begin{aligned} j\omega\begin{Bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} &= -j\omega\begin{bmatrix} K_{23} & K_{24} \\ K_{34} & K_{44} \end{bmatrix}^{-1}\begin{bmatrix} K_{21} & K_{22} \\ K_{41} & K_{42} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \\ \begin{Bmatrix} \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} &= -\begin{bmatrix} K_{23} & K_{24} \\ K_{34} & K_{44} \end{bmatrix}^{-1}\begin{bmatrix} K_{21} & K_{22} \\ K_{41} & K_{42} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \end{aligned} \tag{19}$$

Equations (18) and (19) show that Shape Expansion is done the same way using displacement, velocity, or acceleration frequency domain data, or using displacement time domain data. Of course, to compute stresses and strains, displacement data is still required. Frequency domain velocity or acceleration data can be used directly, but the results must be integrated to displacements in order to calculate stresses and strains.

CONCLUSIONS

A method for calculating structural stresses and strains from experimental ODS data was introduced. The method relies on finite element analysis to calculate the stiffness properties of the structure, from which all unmeasured displacements are then calculated, including in-plane displacements which are not typically measured. In-plane displacements are required to calculate stresses and strains.

Two simple examples were included to illustrate the calculations required to implement this method. Standard finite element analysis generally assumes that the forces are applied at the same DOFs where the experimental ODS data is taken. It was shown that this assumption can be relaxed. By modifying the assumption regarding the known and unknown forces, it was shown by example that a valid solution can still be obtained using the same finite element equations.

The new solution is somewhat restricted in that it requires the inversion of a non-sparse non-symmetric matrix. Therefore, large model sizes may be computationally prohibitive.

This new approach offers two significant advantages. First, it yields more realistic shape interpolations than geometrically based interpolation methods, and it can also be used to extrapolate shapes, which is not possible with geometric interpolation except in the simplest cases. Secondly, by combining experimental ODS data with finite element modeling, this approach provides an alternative for determining structural stress and strain than either finite element analysis alone or the use of experimental strain gages.

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