Using FEA Modes to Scale Experimental Mode Shapes

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ABSTRACT

When Operating Modal Analysis (OMA) is used for finding the modal parameters of a structure, the excitation forces are not measured. Because the forces are not measured, the resulting mode shapes cannot be used in a ***modal model*** because they are not properly scaled to reflect the mass and stiffness properties of the structure.

In a traditional multi-shaker modal survey using sinusoidal signals, the excitation forces are also not measured and the mode shapes are obtained from response only data. Again, these un-scaled shapes cannot be used in a modal model.

Finally, even in an FRF-based impact or shaker where the excitation forces are measured, ***calibrated*** measurements must be made in order to properly scale the mode shapes. Also, a driving point measurement is usually required, which can often be difficult to make, resulting in error prone mode shape scaling.

In this paper, we show how analytical mode shapes obtained from finite element analysis (FEA) can be used to scale experimental mode shapes. It is shown that analytical models having relatively few finite elements in them can yield mode shapes that correlate well with experimental shapes, and are therefore adequate for scaling the experimental shapes. A straightforward least squared error method is introduced for scaling the experimental shapes. Examples are included that illustrate how FEA models of various sizes will still yield accurate results.

INTRODUCTION

Modal Analysis has become the favorite label for what is more accurately called Experimental Modal Analysis (EMA), modal testing, or a modal survey. In an EMA, an experimentalist endeavors to characterize the dynamic behavior of a structure in terms of its modes of vibration, by testing the physical structure. Each mode is defined by its modal frequency, modal damping, and mode shape.

Finite element analysis (FEA) is also done to characterize structural dynamics, by constructing a numerical model that simulates the structure in a computer. FEA also provides the modes of a structure. FEA is analytical, EMA is experimental, and modes are the common ground between the two.

Mode shapes are called "***shapes***" because they are unique in shape, but not in value. That is, the mode shape vector for each mode does not have unique values.

$\{\mathbf{u}_k\} =$ mode shape vector for mode (**k**), (N-vector)

N = number of DOFs of the mode shape vector.

A DOF defines motion at a point on the structure, in a specific direction. Each mode shape can be arbitrarily scaled to any set of values, but the "***shape***" of $\{\mathbf{u}_k\}$ is unique. That is, the ratio of each shape component to any other is unique, but its value is not. A mode shape is also called an ***eigenvector*** for this same reason.

Alternative Dynamic Models

The dynamic characteristics of any mechanical or civil engineering structure can be adequately represented in three different and equivalent ways,

1. A set of ***linear second-order differential equations***, typically used in FEA.
2. An ***FRF matrix model***, using for EMA.
3. A ***modal model***, obtained from either FEA or EMA, with mode shapes properly scaled to preserve the mass and stiffness of the structure. Modal damping, obtained from an EMA, completes the modal mode.

Uses of a Modal Model

A set of mode shapes that is properly scaled to preserve the mass (or inertia) and stiffness (or elastic) properties of a structure is called a ***modal model***. A modal model can be used for different dynamic modeling and simulation studies, including:

1. **FRF Synthesis:** Calculation of FRFs for comparison with experimentally measured FRFs.
2. **Structural Modifications:** Investigating the effects of potential hardware modifications on the modes of a structure.
3. **Forced Response Simulation:** Calculating structural responses due to sinusoidal, random, transient or ambient forcing functions applied to multiple DOFs.
4. **Load Path Analysis:** Calculating the excitation forces that cause multiple measured responses.

5. **Best Modification Search:** Combining Structural Modifications with Forced Response Simulation to find the best places to modify a structure to reduce overall vibration levels.

EMA and FEA are complimentary engineering tools. If an EMA and an FEA on the same structure both yield the same modes, then presumably both must be accurately characterizing its structural dynamics. Both tools are useful for gaining a better understanding of the dynamic behavior of structures, and in particular for understanding and solving resonant vibration problems.

Modal Mass Matrix

The mode shapes of a finite element model are calculated in a manner which "***simultaneously diagonalizes***" both the mass and the stiffness matrices of the model. This is the so-called **orthogonality** property.

When the mass matrix is post-multiplied by the mode shape matrix and pre-multiplied by its transpose matrix, the result is a diagonal matrix, as shown in equation (1). ***This is a definition of modal mass.***

$$[\phi]^t[M][\phi] = \left[\ddots m \ddots\right] \qquad (1)$$

where,

$[\mathbf{M}] =$ mass matrix **(N** by **N)**.

$[\phi] = \left[\{u_1\}\{u_2\}\ldots\{u_M\}\right] =$ mode shape matrix.

t – denotes the transpose.

M = number of modes in the model.

The modal mass of each mode (**k**) is a diagonal element of the modal mass matrix,

Modal mass: $$\mathbf{m_k = \frac{1}{A_k \omega_k}} \qquad \mathbf{k=1,\ldots, M} \quad (2)$$

$\boldsymbol{\omega_k} =$ damped natural frequency of mode(**k**).

$\mathbf{A_k} =$ scaling constant for mode(**k**).

Equation (2) indicates that ***modal masses are arbitrary***, and can also be written in terms of the modal frequency $\boldsymbol{\omega_k}$ and a scaling constant $\mathbf{A_k}$ (*See* [2] & [3] for details of this definition). Since the mode shape values are arbitrary, the modal masses must also be arbitrary in order for equation (1) to be valid.

Scaling Mode Shapes to Unit Modal Masses

One of the common ways to scale mode shapes is so that the modal masses are one (unity). This is called ***unit modal mass*** (UMM) scaling. When a mass matrix $[\mathbf{M}]$ is available, the mode vectors are scaled so that the modal mass matrix is equal to an ***identity matrix***, with diagonal elements equal to one and zeros elsewhere. However, when mode shapes are obtained experimentally, a mass matrix is typically not available for UMM scaling.

EXPERIMENTAL SCALING METHODS

Several methods do exist for UMM scaling of experimental mode shapes, but they all rely on ***calibrated*** experimental measurements. These three methods are discussed in more detail in [1].

Driving Point FRF

This is the most commonly used method for scaling experimental mode shapes. It requires that all FRF measurements made on a structure be properly calibrated. This means that calibrated transducers are used so that each FRF is an accurate measure of the amount of response motion (displacement, velocity, or acceleration) at one DOF per unit of force applied at another DOF.

In addition, for UMM scaling of the mode shapes, a driving point FRF (where the response DOF equals the excitation force DOF) is required. The driving point FRF provides squared mode shape components for each mode, which are then used to scale the mode shapes.

Triangular FRFs

This method doesn't require a driving point FRF measurement, which can often be difficult to make, but does require the measurement of three other particular FRFs. The method is called ***"triangular"*** because the three FRFs form a triangle of elements in the FRF matrix model. This method also allows the measurement of ***diagonal elements*** of the FRF matrix, unlike the traditional measurement of ***one row or column*** of the matrix. This testing approach has advantageous for testing large structures, as explained in [1].

Structural Modification

This method combines the SDM algorithm with a search method to scale the mode shapes. It requires that an actual modification be made to the structure, and that the modal parameters of the modified be measured. Usually, a mass modification that changes all of the modal frequencies is sufficient. The ***"shifted frequencies"*** of the modified structure can be measured with a simple Auto spectrum measurement. Nevertheless, finding the right location (or locations) for the modification can be difficult, especially since the modification must ***affect all of the modes***.

CORRELATING MODE SHAPES

Every structure that vibrates will eventually stop vibrating if all forces causing it to vibrate are removed. Resonant vibration will stop because some kind of damping mechanism, or in most cases a combination of mechanisms, will dissipate the energy from the structure to its surroundings.

The numerous damping mechanisms at work in a real structure cannot be easily modeled, so damping is not included in most FEA models. However, mode shapes can still be obtained from a model with no damping in it. Mass and stiffness cause resonant vibration. Damping only dissipates it. Each analytical shape has a modal frequency associated with it, but no modal damping. On the other hand, an EMA is always done on a structure that has damping in (or around) it. Damping is unavoidable when testing a real structure. Fortunately, most structures, especially those with troublesome resonances, are ***"lightly damped"***. Consequently, the experimental mode shapes from a lightly damped structure will closely approximate analytical shapes from an FEA model with no damping in it.

Another disparity between analytical and experimental mode shapes is the number of DOFs in the mode shapes. A typical EMA may include hundreds of measurements, which will yield experimental mode shapes with hundreds of DOFs in them. Alternatively, an FEA typically yields analytical mode shapes with thousands, even millions of DOFs in them.

Although the experimental shapes will usually have far fewer DOFs than the analytical shapes, both shapes can be compared at ***all DOFs that are common*** to both of them. Two different numerical methods have become popular for quantitatively comparing mode shapes, the Orthogonality check and the Modal Assurance Criterion (or MAC).

Orthogonality Check

The orthogonality check was first adopted in the early days when EMA was done to confirm FEA models. An orthogonality check attempts to "diagonalize" the mass matrix of the FEA model, using both the experimental and analytical mode shapes. Instead of using analytical shapes to pre- and post-multiply the mass matrix in equation (1), experimental mode shapes are used on one side, and analytical shapes on the other of the matrix triple product.

The main difficulty with the orthogonality check is that the mass matrix and analytical shapes typically have many more DOFs in them than the experimental shapes. Therefore, a non-trivial step of reducing the size of the mass matrix to match the size of the experimental mode shapes is required. There are a number of ways for reducing the size of the mass matrix, but the details will not be discussed here. Nevertheless, the result of an orthogonality check is this;

Mass Matrix Orthogonality: If the experimental mode shapes are the same as the analytical mode shapes at DOFs that are common between them, the orthogonality check will yield a diagonal mass matrix.

The converse of orthogonality is also true. If the orthogonality check does not result in a diagonal matrix, then the analytical and experimental mode shapes are different.

Modal Assurance Criterion (MAC)

A key advantage of the MAC method over the orthogonality check in that it only requires the mode shapes themselves. By eliminating the mass matrix from the calculation, one possible source of error is removed from the shape comparison. Like the orthogonality check, MAC yields a **value of "1"** when an experimental and analytical shape are the same, and a value **less than "1"** when they are different.

Can EMA and FEA Yield the Same Mode Shapes?

From a theoretical point of view, experimental mode shapes should match analytical shapes at all common DOFs, since both characterize the dynamics of the same structure. However, there are a number of practical reasons why experimental mode shapes won't match analytical shapes.

One significant reason is that the boundary conditions may be different between the EMA and FEA. If the boundary conditions are different, the mode shapes will be different. For example, the modes of a cantilever beam are clearly different from those of a free-free beam.

It is often difficult to reproduce the same boundary conditions in an EMA that were used during construction of the FEA model. Conversely, the flexibility of floors, walls, platforms, mounts, and all types of boundaries which may be assumed as rigid in an FEA, may significantly affect the modes of the real structure. This is where the complementary nature of these two tools becomes important.

Since they are complimentary, it makes sense to take advantage of the strengths of both FEA and EMA.

"The ideal modal model might be one which combines ***experimental frequencies and damping*** with ***analytical mode shapes***, once the experimental and analytical shapes have been correlated at common DOFs."

MODES OF A BEAM STRUCTURE

The experimental modes of the beam structure shown in Figure 1 will be compared with its analytical mode shapes. The beam was constructed out of three 3/8 inch thick aluminum plates fastened together with cap screws. The overall dimensions of the beam are 12 in. long by 6 in. wide by 4.5 in. high.

The experimental modes were obtained from a set of 99 FRFs which were acquired during an impact test of the beam structure. During the test, the structure was impacted at the same DOF, and a roving tri-axial accelerometer was used to measure its 3D response at 33 points. The resulting experimental mode shapes had 3 DOFs per point, for a total of 99 DOFs.

For a first comparison of shapes, analytical modes were obtained from an FEA model with 161 points and 132 Quad plate elements, as shown in Figure 2. The analytical mode shapes have 6 DOFs (3 translational and 3 rotational) per point, for a total of 996 DOFs.

The beam structure was tested while resting on a foam rubber base, which closely approximates a free-free boundary condition. The modes of the FEA model were also calculated using free-free boundaries.

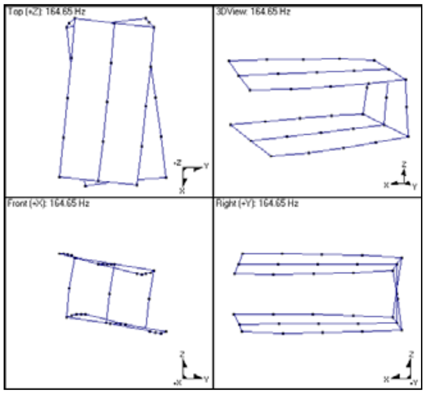

Figure 1 A. 164 Hz Mode.

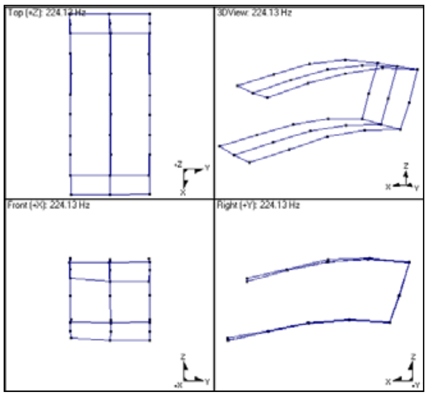

Figure 1 B. 224 Hz Mode.

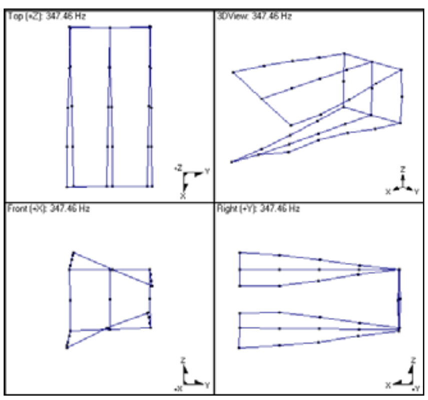

Figure 1 C. 347 Hz.

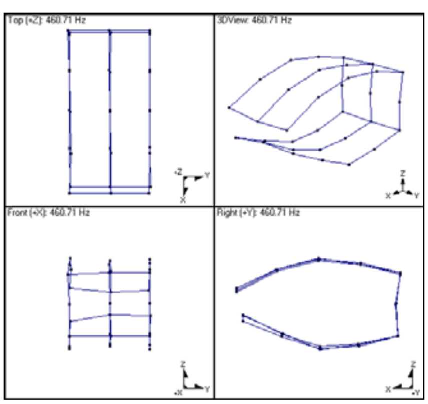

Figure 1 D. 460 Hz Mode.

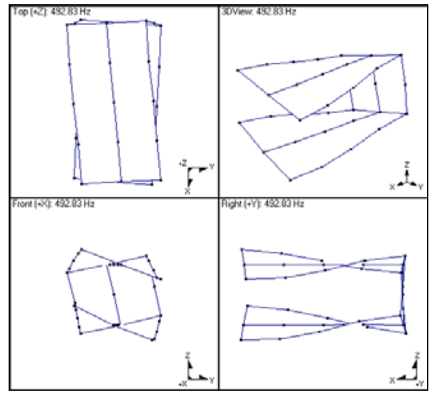

Figure 1 E. 492 Hz Mode.

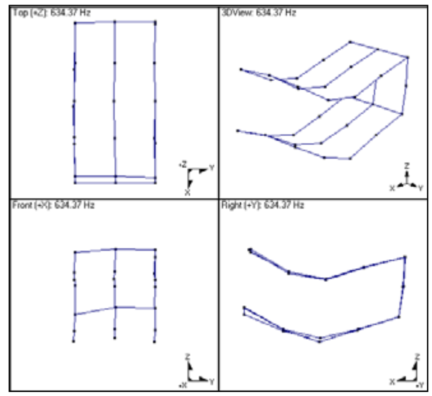

Figure 1 F. 634 Hz Mode.

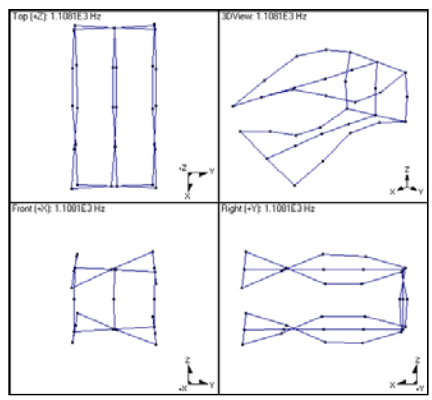

Figure 1 G. 1108 Hz Mode.

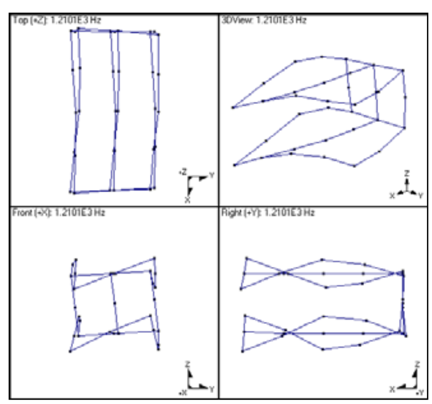

Figure 1 H. 1210 Hz Mode.

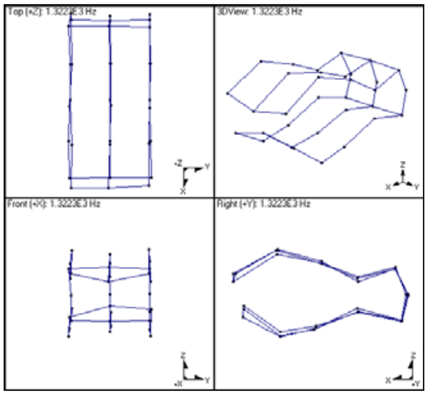

Figure 1 I. 1322 Hz Mode.

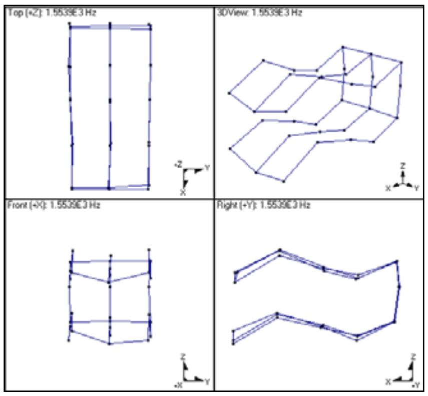

Figure 1 J. 1553 Hz Mode.

The analytical shapes have about 10 times the number of DOFs as the experimental shapes, but the analytical shapes do have translational DOFs at the same 33 points as the experimental shapes. These common DOFs were then used for comparing shapes, and for scaling the experimental mode shapes to UMM shapes.

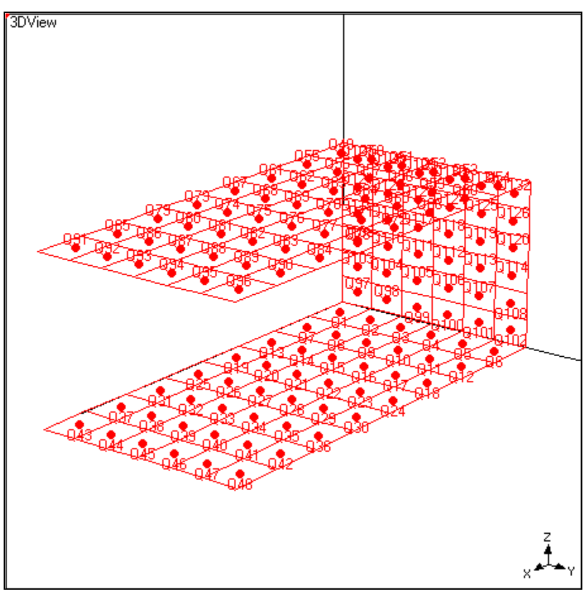

Figure 2. FEA Model with 132 Quad Plate Elements.

Experimental & Analytical Shape MAC Values

The FEA model was solved for its first (lowest frequency) 20 modes, and 10 of those mode shapes were matched with the 10 experimental mode shapes. Only translational DOFs of the analytical shapes at the same 33 points as the experimental shapes were used for comparison. Table 1 lists the analytical and experimental modal frequencies, as well as the MAC values between the 3D shapes, with shape components in the X, Y & Z direction at each of the 33 test points.

Mode	Analytical Frequency (Hz)	Experimental Frequency (Hz)	3D MAC
1	151.80	164.65	0.957
2	212.38	224.13	0.958
3	320.39	347.46	0.950
4	419.17	460.71	0.933
5	462.64	492.83	0.951
6	593.96	634.37	0.938
7	1.0377E3	1.1081E3	0.905
8	1.1327E3	1.2101E3	0.895
9	1.1885E3	1.3223E3	0.852
10	1.4093E3	1.5539E3	0.829

Table 1. Analytical vs. Experimental Shape Comparison.

Table 2 lists the MAC values using shape components in the X, Y and Z directions only.

Mode	X Only MAC	Y Only MAC	Z Only MAC
1	0.979	0.940	0.975
2	0.976	0.237	0.947
3	0.335	0.003	0.985
4	0.184	0.077	0.968
5	0.876	0.875	0.982
6	0.636	0.013	0.973
7	0.275	0.000	0.961
8	0.252	0.648	0.960
9	0.449	0.036	0.928
10	0.056	0.010	0.928

Table 2. MAC Values for X, Y, Z Directions Only.

Table 1 shows that even though the analytical modal frequencies are quite different from the experimental frequencies, the mode shapes are very similar (.90 and above) for all but the highest frequencies.

Table 2 reveals that when only the Z directions of the mode shapes are used, the analytical and experimental shapes agree even more closely than the 3D shapes. Additionally, the X and Y Only MAC values indicate that the shapes don't agree very well in those directions. This is best explained by that fact that the ***dominate motion*** of all of the shapes (except Mode #1) is in the Z direction.

Since motion in the X & Y directions is significantly less the Z direction, it can be assumed that there is error in the X & Y directions of the experimental shapes. Nevertheless, we can conclude that there is a ***strong correlation*** between ***all 10 analytical and experimental shapes*** in the Z direction, where the motion is dominant.

COMPARING FRFs

Another way of comparing analytical and experimental results is to synthesize FRFs using the modal parameters, and overlay the synthesized FRFs on the experimental FRFs. To do this, we will use a ***hybrid modal model***.

Hybrid Modal Model: This model consists of the ***experimental modal frequencies & damping*** for each mode, together with ***analytical mode shapes*** that are ***UMM scaled.***

Figure 3 shows two examples of synthesized FRFs using the hybrid modal model overlaid on experimental FRFs in Bode format. In both cases, the synthesized FRFs are in close agreement with the experimental FRFs.

SMALLER SIZED FEA MODEL

To see how well the analytical shapes from a smaller sized FEA model compare with the experimental shapes, we built a second model with Quad plate elements ***only between the 33 test points on the experimental model***. Figure 4 shows the model, with 20 Quad plate elements in it.

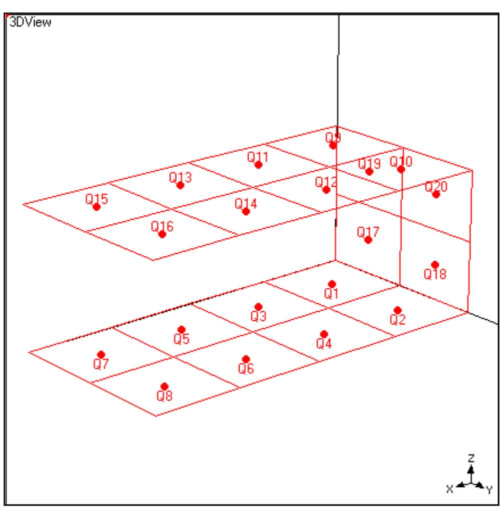

Figure 4. FEA Model with 20 Quad Plate Elements.

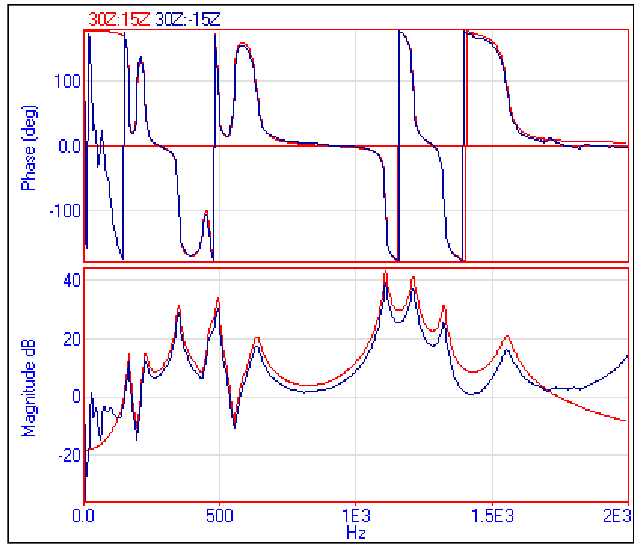

Figure 3A Overlaid Synthesized and Measured FRFs Between Top & Bottom Plate.

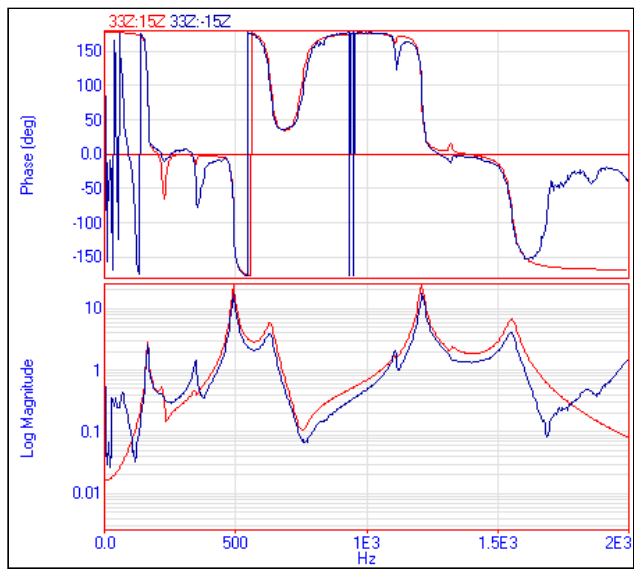

Figure 3B Overlaid Synthesized and Measured FRFs Between Top & Back Plate.

We then solved for the modes of this model and compared them with the experimental shapes in the Z direction only. Those results are shown in Table 3.

For this smaller model, there is an even greater disparity between the analytical and experimental modal frequencies. However, the Z direction MAC values indicate that the analytical and experimental mode shapes are still in very close agreement.

An accurate hybrid modal model can again be constructed by combining the experimental frequencies and damping with the UMM scaled mode shapes from the smaller FEA model.

Mode	Analytical Frequency (Hz)	Experimental Frequency (Hz)	Z Only MAC
1	138.02	164.65	0.966
2	203.95	224.13	0.940
3	265.68	347.46	0.984
4	387.33	460.71	0.921
5	397.64	492.83	0.913
6	546.45	634.37	0.966
7	832.14	1.1081E3	0.956
8	913.14	1.2101E3	0.946
9	1.05E3	1.3223E3	0.928
10	1.2148E3	1.5539E3	0.882

Table 2. Shape Comparison from Smaller FEA Model.

MODE SHAPE SCALING

Having seen that it is relatively straightforward to obtain analytical shapes that correlated well with experimental shapes, we now turn to the topic of this paper, namely scaling a set of experimental shapes to UMM using a set of analytical shapes. Scaling assumes that each experimental mode shape has already been correlated with an analytical shape that is UMM scaled.

The scale factor $SF_e(k)$ required to scale each experimental shape using an analytical shape is the solution to the equation,

$$\{U_e(k)\}SF_e(k) = \{U_a(k)\} \tag{1}$$

where:

k = 1, 2 ,…, modes

$\{U_e(k)\}=$ experimental mode shape (N-vector)

$\{U_a(k)\}=$ analytical mode shape (N-vector)

N = number of common DOFs

The analytical mode shape vector $\{U_a(k)\}$ is real valued, and for that reason is called a ***normal*** mode shape. The experimental mode shape $\{U_e(k)\}$ is complex valued, with each shape component having real and imaginary parts. Therefore, the scale factor $SF_e(k)$ is also complex.

Equation (1) is a set of equations, one for each DOF that is common between the analytical and experimental shapes. The ***least squared error solution*** to equation (1) is,

$$SF_e(k) = \frac{\{U_e(k)\}^T\{U_a(k)\}}{\left|\{U_e(k)\}\right|^2} \tag{2}$$

where:

$T =$ transposed conjugate of the complex vector.

Equation (2) provides the scale factor which scales each experimental shape $\{U_e(k)\}$ to UMM.

Equation (1) can also be re-written so that $\{U_a(k)\}$ is scaled instead of $\{U_e(k)\}$,

$$\{U_a(k)\}SF_a(k) = \{U_e(k)\} \tag{3}$$

where:

$SF_a(k) =$ scale factor for scaling $\{U_a(k)\}$ to $\{U_e(k)\}$

The ***least squared error solution*** to equation (3) is,

$$SF_a(k) = \frac{\{U_a(k)\}^T\{U_e(k)\}}{\left|\{U_a(k)\}\right|^2} \tag{4}$$

By comparing equations (1) and (3), it is clear that,

$$SF_a(k) = \frac{1}{SF_e(k)} \tag{5}$$

The product of these two scale factors is,

$$SF_e(k)\,SF_a(k) = \frac{\left|\{U_e(k)\}^T\{U_a(k)\}\right|^2}{\left|\{U_e(k)\}\right|^2\left|\{U_a(k)\}\right|^2} \tag{6}$$

This is the formula for the MAC value between two shapes. In other words, the product of the two mode shape scale factors is a ***number between "0" and "1"***.

If the two mode shapes have the same ***"shape",*** the MAC value will be ***"1"***, and the two scale factors are ***inverses*** of one another according the equation (5). If, however, the two shapes are different, the two scale factors will not be inverses of one another. In the worse case, if the two shapes are ***"orthogonal"*** to one another, both equations (2) and (4) give scale factors of ***"0".*** Clearly, both equations (2) and (4) say that when two shapes are ***different*** from one another, one shape cannot be scaled to match the other.

To test equations (2) and (4), the experimental mode shapes extracted from a set of FRFs measured on the beam structure in Figure (1) were UMM scaled using the analytical shapes for the beam. The experimental mode shapes are called ***Residue mode shapes*** since they were obtained by curve fitting the set of FRFs with a specific Reference DOF (impact point and direction) of 15Z. These shapes are clearly not UMM mode shapes.

Mode	$SF_e(k)$	$\frac{1}{SF_a(k)}$	Product of Scale Factors
1	(0.0432,-176)	(0.0452,-176)	0.955
2	(0.0281,-176)	(0.0293,-176)	0.959
3	(0.00984,3.73)	(0.0104,3.73)	0.946
4	(0.0123,0.11)	(0.0132,0.11)	0.932
5	(0.00807,-0.32)	(0.00849,-0.32)	0.950
6	(0.0103,180)	(0.0109,180)	0.945
7	(0.00291,1.63)	(0.00321,1.63)	0.906
8	(0.00305,179)	(0.00341,179)	0.894
9	(0.00693,-178)	(0.00813,-178)	0.852
10	(0.00744,179)	(0.00897,179)	0.829

Table 3. Scale Factors between Experimental and Analytical Mode Shapes.

The scale factors $SF_e(k)$ and $SF_a(k)$ for the 10 shapes are listed in Table 3. Notice that the ***product of the scale factors closely matches the MAC values*** in Table 1.

The scale factors $SF_e(k)$ from Table 3 were used to scale the experimental shapes, and FRFs were synthesized using the ***experimental modal model*** for comparison with corresponding FRF measurements. A typical synthesized FRF is overlaid on a measured FRFs in Figure 4.

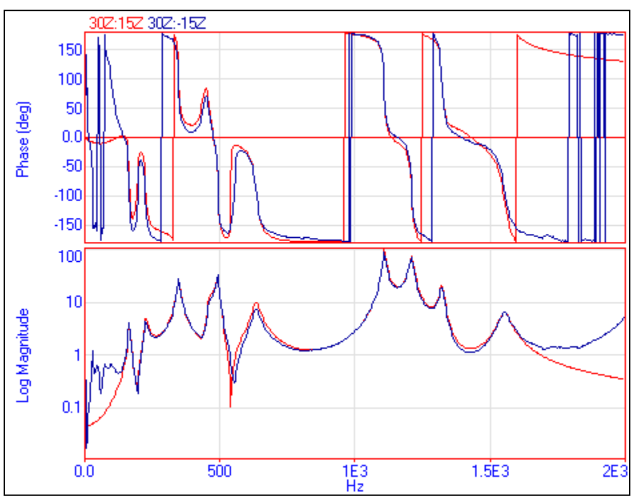

Figure 4. Overlaid Synthesized and Measured FRFs Between Top & Back Plate.

CONCLUSIONS

Mode shape scaling is important if a modal model is to be used for further modeling and simulation studies. While EMA and FEA can both provide modal parameters, the combination of experimental frequencies and damping with analytical mode shapes into a ***hybrid modal model*** provides the best use of both the EMA and FEA data.

It was shown that a very small FEA model can provide mode shapes that correlate well with experimental shapes, even though the modal frequencies are substantially different. The accuracy of the hybrid modal model was verified by overlaying synthesized and experimental FRFs.

A formula for scaling experimental shapes from a set of FEA shapes was presented, and its close relationship to the MAC formula was also shown. Finally, a set of experimental residue mode shapes was UMM scaled using a set of FEA mode shapes. The accuracy of the resulting ***experimental modal model*** was again verified by overlaying synthesized and experimental FRFs.

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