Matrix formulation of multiple and partial coherence

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The conventional theory of coherence is essentially a scalar theory in the sense that coherence values are generally expressed as scalars obtained from the elements of various auto- and cross-spectral matrices. Furthermore, the theory is generally described for the single-output case, and for the case where only one undesired interfering input is assumed. In this paper, we present a general matrix formulation of coherence for the multiple-input and output case, including an arbitrary number of interfering inputs. We show that for n inputs, there are $2^n - 1$ different output coherence matrices.

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INTRODUCTION

In linear systems having multiple inputs and multiple outputs, there are often various interrelationships among the inputs, as well as several signal paths between each input and each output. In addition, there are usually unrelated sources of noise or interference that contaminate the inputs and outputs.

Let's illustrate the essence of the problem with a practical example. Suppose that we want to study the noise environment in the interior of an automobile. We decide to monitor the noise using two microphones, one located in the front-seat area and one in the rear-seat area. We are particularly interested in the noise-reduction properties of the suspension system, so we instrument the vehicle with accelerometers on the wheel axles. However, there are noises from the engine that tend to contaminate our measurements, so we also add transducers to monitor these sources in the engine compartment. Finally, there are wind noises due to turbulent airflow that we cannot easily measure, but nevertheless are very noticeable.

We actually want to know the contribution at each microphone from each wheel source, with all engine and wind noises removed. Now, the wind noises are completely unrelated (incoherent) to both the road and engine noises, so this contribution can readily be eliminated from the estimate of the transfer characteristics of the suspension system by averaging techniques. However, the engine noises will travel through the vehicle frame to each wheel axle, and will be measured by those wheel transducers, as though these contributions actually came from the road, so we want to remove these interfering engine noises from each wheel transducer. Then we can follow the "residual" wheel-transducer signals through the suspension system and to the monitoring system.

The concept of a "residual" input obtained by removing unwanted signals is very useful, and leads directly to the partial coherence concept. We do this removal using a least-squared error technique, as will be described later.

The interrelations among the various inputs are completely described by an input-power matrix P, and the transfer characteristics between the inputs and outputs are described by the transfer matrix H. Although these two matrices determine the output power HPH^T that originates from the defined inputs, there are generally other output contributions from various noise sources. These noise components are described by the noise power matrix N, giving a total output-power matrix R. The coherence matrix Γ times R gives that portion of R that originates from the inputs described by P. The unit identity matrix I minus Γ, multiplied by R gives the noise matrix N. The coherence matrices are either multiple or partial, depending upon whether all inputs are considered to be of interest, or whether some of the inputs are considered to be contaminating, and are thus removed from the remaining inputs.

The ordinary scalar coherence function (for one input and one output) gives the proportion of the total output power (at a particular output) that *seems* to originate from a particular input. This is simply the ratio of the diagonal elements of the coherent output power [given by HPH^T; see Eq. (22)] and the total output power R. However, for the multiple-input case, there are relationships among the various inputs, as well as many paths from each input to a particular output, so the ordinary coherence becomes difficult to interpret. The situation is further complicated when we have multiple outputs, and we include the possibility that the various interfering noise sources at each output may be related. A matrix formulation of coherence becomes essential in this situation.

I. LINEAR DEPENDENCE BETWEEN TWO VECTORS

We can best lay the theoretical groundwork by describing the way that two arbitrary vectors can be linearly related. Suppose that we have any two vectors, U and V (not necessarily the same number of elements), and we would like to know how much of V is linearly related to U, and how much is incoherent noise, which we will denote by W. We postulate that

$$V = TU + W , \tag{1}$$

where T is some transformation matrix that describes the relationship between U and V. We define W to be incoherent with U, so

$$E[WU^T] = 0 , \tag{2}$$

where E denotes the expected value of the product (obtained by averaging an infinite number of estimates to-

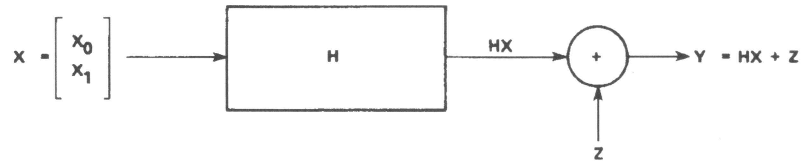

FIG. 1. Multiple input-output model.

gether), and the T superscript denotes a conjugate transpose operation. These vectors and matrices will generally be complex functions of the Laplace variable s.

Let's postmultiply Eq. (1) by U^T, to obtain

$$E[VU^T] = TE[UU^T] + E[WU^T] = TE[UU^T] . \tag{3}$$

From this relation, we can solve for T, giving

$$T = E[VU^T]E[UU^T]^{-1} , \tag{4}$$

provided the inverse of $E[UU^T]$ exists. A necessary and sufficient condition for this inverse to exist is that none of the elements of U be completely coherent with one another. This will insure that all columns (or rows) of $E[UU^T]$ are linearly independent.

It is easy to show that Eq. (2) insures that TU is the best estimate of V in a least-squares sense, but we will not digress to derive this relationship. We call $W = V - TU$ the *residual* vector.

Analogous to the "power" matrix $E[UU^T]$, we can also define the "power" associated with the V vector as

$$E[VV^T] = E[(TU + W)(TU + W)^T]$$
$$= TE[UU^T]T^T + E[WW^T] . \tag{5}$$

We see that the power associated with V comprises the part

$$TE[UU^T]T^T = E[(TU)(TU)^T]$$

that is associated with the part of V coherent with U, and the part $E[WW^T]$ associated with the incoherent residual vector W.

We define a coherence matrix Γ as follows:

$$\Gamma E[VV^T] = TE[UU^T]T^T$$
$$= E[VU^T]E[UU^T]^{-1}E[VU^T]^T , \tag{6}$$

so, from Eq. (5) we get

$$(I - \Gamma)E[VV^T] = E[WW^T] , \tag{7}$$

giving

$$\Gamma = I - E[WW^T]E[VV^T]^{-1}$$
$$= E[VU^T]E[UU^T]^{-1}E[VU^T]^T E[VV^T]^{-1} . \tag{8}$$

Note that Γ times the total power gives the part coherent with U, and $I - \Gamma$ times the total power gives the incoherent part. For future reference, we can rewrite Eq. (7) as

$$E[WW^T] = (I - \Gamma)E[VV^T]$$
$$= E[VV^T] - E[VU^T]E[UU^T]^{-1}E[VU^T]^T . \tag{9}$$

There is an interesting alternate way of obtaining this residual-power matrix as follows: Let's define a composite vector $\begin{bmatrix} U \\ V \end{bmatrix}$, and write the power matrix ρ for this new vector. We obtain

$$\rho = E\left[\begin{bmatrix} U \\ V \end{bmatrix}\begin{bmatrix} U^T & V^T \end{bmatrix}\right] = \begin{bmatrix} E[UU^T] & E[VU^T]^T \\ E[VU^T] & E[VV^T] \end{bmatrix} . \tag{10}$$

Now, if we invert this matrix, we obtain

$$\rho^{-1} = \begin{bmatrix} \cdots & \cdots \\ \cdots & E[WW^T]^{-1} \end{bmatrix} , \tag{11}$$

where we have ignored all terms except those that occupy the *same position* that $E[VV^T]$ occupied in the original ρ matrix. Using these submatrices, we can calculate Γ in the form given by Eq. (8):

$$\Gamma = I - E[WW^T]E[VV^T]^{-1} = I - (E[VV^T]E[WW^T]^{-1})^{-1} . \tag{12}$$

We next apply this general theory of linear dependence between two vectors to the multiple input–output system, defining the residual input vector, and the concepts of input coherence, partial coherence, and multiple coherence.

II. THE MULTIPLE INPUT-OUTPUT SYSTEM

Figure 1 illustrates the linear system model that we plan to discuss, where we have subdivided the input vector X into two parts, X_0 and X_1. We call the X_0 elements "desired" inputs, and the X_1 elements "interfering" inputs. We emphasize that X_1 may be zero. The system transfer matrix is H, and the contaminating noise vector is Z, where we define Z to be incoherent with X, expressed by

$$E[ZX^T] = 0 . \tag{13}$$

This system model is obtained by first writing the various simultaneous time-domain equations describing the physical system, and then taking the Laplace transform. Thus, all of these vector and matrix elements are functions of the Laplace variable s. When we actually estimate these quantities, we use the Fourier transform in which s is replaced by $i2\pi f$ (f is frequency in hertz).

For the moment, let's ignore the partitioning of the input X vector, so the output Y is simply

$$Y = HX + Z . \tag{14}$$

Postmultiply by X^T and take expected values, to get

$$E[YX^T] = HE[XX^T] + E[ZX^T] = HE[XX^T] . \tag{15}$$

To shorten the notation, we define P, Q, R, and N matrices as follows:

$$P = E[XX^T]\text{, input-power matrix} \quad (16)$$

$$Q = E[YX^T]\text{, cross-power matrix} \quad (17)$$

$$R = E[YY^T]\text{, output-power matrix} \quad (18)$$

$$N = E[ZZ^T]\text{, noise-power matrix} \quad (19)$$

In these terms, we can rewrite Eq. (15) as

$$Q = HP\ , \quad (20)$$

from which we obtain

$$H = QP^{-1} \quad (21)$$

for the transfer matrix. None of the inputs can be completely coherent, or else P will be singular. Forming the output-power matrix $R = E[YY^T]$ gives

$$R = HPH^T + N\ . \quad (22)$$

It is apparent that HPH^T is the part of R that is coherent with the input X, and N is the incoherent noise contribution.

We define the *multiple* coherence matrix Γ_m by the expression

$$\Gamma_m R = HPH^T = R - N\ , \quad (23)$$

so

$$(I - \Gamma_m) R = N \quad (24)$$

and

$$\Gamma_m = I - NR^{-1} = HPH^T R^{-1}\ . \quad (25)$$

In actual practice, we are seldom particulary interested in the elements of Γ_m, but rather are more concerned with the two parts of R, corresponding to the coherent part $\Gamma_m R$, and the incoherent part $N = (I - \Gamma_m)R$. We emphasize this point of view by writing

$$R = \Gamma_m R + (I - \Gamma_m) R = \Gamma_m R + N\ . \quad (26)$$

Now, we turn our attention to the input problem in which X is partitioned into the desired part X_0, and the interfering part X_1. Our strategy is to remove the interfering X_1 components from the desired X_0 components in a least-squared manner, to obtain a *residual* input vector, which we will call $\tilde{X}$. Referring to the previous section describing the linear dependence between two vectors, we can identify X_1 with U, X_0 with V, and $\tilde{X}$ with W. We can immediately write the least-squares transformation matrix T [from Eq. (4)] as

$$T = E[X_0 X_1^T]\, E[X_1 X_1^T]^{-1} \quad (27)$$

so, the *residual* input vector $\tilde{X}$ becomes [from Eq. (1)]

$$\tilde{X} = X_0 - TX_1\ . \quad (28)$$

Let's reemphasize that $\tilde{X}$ is the remnant of the desired input vector X_0, after the least-squares removal of the interfering input vector X_1. It should be apparent that ere are numerous possible $\tilde{X}$ vectors, depending upon the partitioning of X. For example, if there are n elements in X, ν of which are in X_1 and $n - \nu$ of which are in X_0, then there are $n!/\nu!(n-\nu)! = \binom{n}{\nu}$ different possible combinations of desired and interfering inputs, and hence $\binom{n}{\nu}$ different residual input vectors $\tilde{X}$. If we sum these binomial coefficients over ν, from 0 to $n - 1$, we obtain $2^n - 1$ different possibilities. One of these possibilities is for $\nu = 0$, implying no interfering inputs, and in this case $\tilde{X} = X_0 = X$. We must exclude the case for $\nu = n$, because this implies that *all* inputs are interference.

Continuing our study of the residual inputs, we can define an input coherence matrix analogous to Eq. (6) by the expression

$$\Gamma_i E[X_0 X_0^T] = TE[X_1 X_1^T]\, T^T$$
$$= E[X_0 X_1^T]\, E[X_1 X_1^T]^{-1}\, E[X_0 X_1^T]^T \quad (29)$$

Again, to simplify the notation, we will define

$$P_0 = E[X_0 X_0^T]\text{, desired input power} \quad (30)$$

$$P_1 = E[X_1 X_1^T]\text{, interfering input power} \quad (31)$$

$$\tilde{P} = E[\tilde{X}\tilde{X}^T]\text{, residual input power} \quad (32)$$

$$\tilde{Q} = E[X_0 X_1^T]\text{, residual cross power.} \quad (33)$$

From Eq. (7), we can write (29) as

$$\Gamma_i P_0 = TP_1 T^T = \tilde{Q} P_1^{-1} \tilde{Q}^T = P_0 - \tilde{P}\ , \quad (34)$$

or

$$(I - \Gamma_i) P_0 = \tilde{P} = P_0 - \tilde{Q} P_1^{-1} \tilde{Q}^T\ , \quad (35)$$

giving

$$\Gamma_i = I - \tilde{P} P_0^{-1} = \tilde{Q} P_1^{-1} \tilde{Q}^T P_0^{-1}\ . \quad (36)$$

Also, Eq. (27) for T becomes

$$T = \tilde{Q} P_1^{-1}\ . \quad (37)$$

We see from Eq. (35), that the residual input power $\tilde{P}$ is obtained from the desired input power P_0 by subtracting the greatest possible contribution caused by the interfering inputs, $\Gamma_i P_0$. The P_0 matrix can be separated into two parts as

$$P_0 = \Gamma_i P_0 + (I - \Gamma_i) P_0 = \Gamma_i P_0 + \tilde{P}\ , \quad (38)$$

where $\Gamma_i P_0$ is the interfering input power and $(I - \Gamma_i)P_0$ is the residual input power.

This concept of input coherence does not seem to appear in the current literature, but it is a useful way to visualize the residual input approach. As with multiple coherence, the elements of Γ_i are not as important as the product $\Gamma_i P_0$, or the residual power matrix $\tilde{P}$.

We conclude our discussion by defining a partial-coherence matrix (for each $\tilde{X}$). Figure 2 shows the same linear system as Fig. 1, except the input vector is the residual $\tilde{X}$, instead of the complete X vector. Thus, the residual part of the output $\tilde{R}$ is obtained in place of R.

III. SYSTEM MODEL WITH RESIDUAL INPUT

The residual output vector $\tilde{Y}$ is given by

$$\tilde{Y} = \tilde{H}\tilde{X} + Z\ , \quad (39)$$

where $\tilde{H}$ comprises those columns of H corresponding to the elements of $\tilde{X}$ (or X_0). The output residual power $\tilde{R}$ is

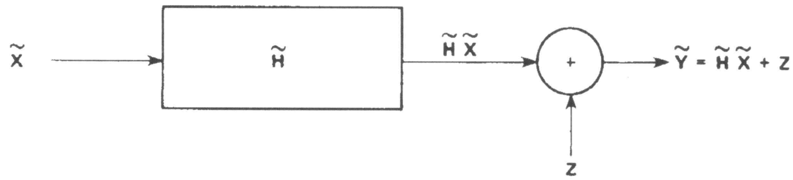

FIG. 2. System model with residual input

$$\tilde{R} = E[\tilde{Y}\tilde{Y}^T] = \tilde{H}\tilde{P}\tilde{H}^T + N \ . \tag{40}$$

The partial-coherence matrix for this particular $\tilde{X}$ input vector is defined by the expression

$$\Gamma_p \tilde{R} = \tilde{H}\tilde{P}\tilde{H}^T = \tilde{R} - N \ , \tag{41}$$

so

$$(I - \Gamma_p)\tilde{R} = N \tag{42}$$

and

$$\Gamma_p = I - N\tilde{R}^{-1} = \tilde{H}\tilde{P}\tilde{H}^T\tilde{R}^{-1} \ . \tag{43}$$

Thus, we can write the two parts of the residual output power as

$$\tilde{R} = \Gamma_p\tilde{R} + (I - \Gamma_p)\tilde{R} = \Gamma_p\tilde{R} + N \tag{44}$$

where $\Gamma_p\tilde{R}$ is the coherent part with residual input $\tilde{X}$ and $(I - \Gamma_p)\tilde{R}$ is the incoherent part.

Finally, we can include the output contribution from the interfering inputs to obtain a total output vector

$$Y = \tilde{H}\tilde{X} + (HX - \tilde{H}\tilde{X}) + Z \tag{45}$$

and a corresponding total output-power matrix

$$R = (\tilde{R} - N) + (R - \tilde{R}) + N \ ,$$

where $(\tilde{R} - N)$ is coherent with $\tilde{X}$ (and part of X_0), but *incoherent* with X_1; $(R - \tilde{R})$ is incoherent with $\tilde{X}$, but coherent with X_1 (and part of X_0); and N is incoherent with all inputs.

Before we summarize these results, we will work out an example involving three related inputs, and a pair of outputs, using two different partitions of the input vector.

IV. EXAMPLE

In the Introduction, we gave an example where two microphones might be used to determine the effects of an automobile suspension system in reducing the transmission of road noises into the passenger compartment. We will calculate some of the coherence matrices for this example, using some assumed input relations, and some arbitrary transfer matrix elements.

The three input-transducer signals are X_a, X_b, and X_c, related to three mutually incoherent sources ξ_1, ξ_2, ξ_3, which might represent road noises from two wheels, and engine noise, etc. The signals from a pair of monitoring microphones in the passenger compartment are represented by Y_1 and Y_2, while the contributions from wind noise are denoted by Z_1 and Z_2, respectively. The transfer matrix H includes the suspension system, as well as the various paths through the frame to the passenger area. (See Fig. 3.) The input vector is

$$X = \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} \ ,$$

and the noise vector is

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} ,$$

giving an output vector

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} .$$

Let's assume that the inputs are related in the following way:

$$X_a = \xi_1 + \xi_2 + \xi_3 \ , \qquad X_b = \xi_2 + \xi_3 \ , \qquad X_c = \xi_3 \ , \qquad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

where the ξ's are mutually incoherent sources. We will choose

$$E[\xi\xi^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \ ,$$

along with

$$H = \begin{bmatrix} 5 & -i3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} .$$

These are completely arbitrary numbers, chosen simply to illustrate the previous theory. First, let's calculate the input-power matrix P:

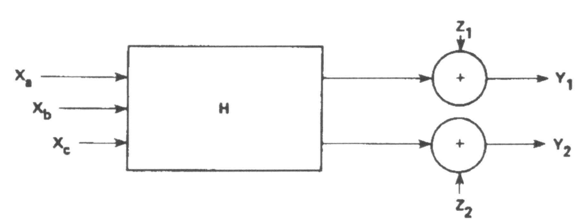

FIG. 3. 3-input, 2-output system.

$$P = E[XX^T] = \begin{bmatrix} 6 & 5 & 3 \\ 5 & 5 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Since the total output-power matrix is $R = HPH^T + N$, we have

$$HPH^T = \begin{bmatrix} 195 & 55 - i33 \\ 55 + i33 & 29 \end{bmatrix},$$

giving,

$$R = \begin{bmatrix} 196 & 55 - i33 \\ 55 + i33 & 33 \end{bmatrix} .$$

We can invert both the input- and output-power matrices, giving

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{5}{6} \end{bmatrix}$$

and

$$R^{-1} = \frac{1}{2354} \begin{bmatrix} 33 & -55 + i33 \\ -55 - i33 & 196 \end{bmatrix}.$$

The coherent output power $\Gamma_m R$ is given by

$$\Gamma_m R = R - N = HPH^T = \begin{bmatrix} 195 & 55 - i33 \\ 55 + i33 & 29 \end{bmatrix},$$

so, we can write the multiple-coherence matrix as

$$\Gamma_m = \frac{1}{2354} \begin{bmatrix} 2321 & 55 - i33 \\ 220 + i132 & 1570 \end{bmatrix} .$$

Note that the elements of Γ_m are difficult to interpret, whereas the elements of $\Gamma_m R$ have a direct physical meaning.

For case I, let's assume that ξ_1 is the desired input signal, and ξ_2 and ξ_3 are interfering inputs. We have managed to locate transducers to monitor ξ_2 and ξ_3 via X_b and X_c, so we can remove these signals from X_a. Thus, we define $X_0 = X_a$ and $X_1 = [^{X_b}_{X_c}]$. Using Eqs. (30)–(33), we find that

$$P_0 = 6, \quad P_1 = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}, \quad P_1^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{6} \end{bmatrix},$$

$$\tilde{Q} = [5 \quad 3],$$

so

$$T = \tilde{Q} P_1^{-1} = [1 \quad 0] .$$

The residual-input vector thus becomes

$$\tilde{X} = X_0 - TX_1 = X_a - X_b = \xi_1 .$$

We expect this result, because T has been chosen to eliminate all possible contributions from X_b and X_c to X_a. Obviously, the residual input power is

$$\tilde{P} = E[\xi_1 \xi_1^T] = 1 .$$

The input coherence is [from Eq. (36)]

$$\Gamma_i = I - \tilde{P} P_0^{-1} = \tfrac{5}{6},$$

while

$$\Gamma_i P_0 = 5 , \quad \text{and} \quad (I - \Gamma_i) P_0 = 1 .$$

Thus, out of a total input power of six units at input X_a, five of these units are interfering signals, and only one unit is associated with the residual input.

To determine the partial coherence, we need the $\tilde{H}$ and $\tilde{R}$ matrices. $\tilde{H}$ simply comprises the first column of H, corresponding to $X_0 = X_a$, so

$$\tilde{H} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

and

$$\tilde{H}\tilde{P}\tilde{H}^T = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix},$$

giving

$$\tilde{R} = \tilde{H}\tilde{P}\tilde{H}^T + N = \begin{bmatrix} 26 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$\Gamma_p \tilde{R} = \tilde{H}\tilde{P}\tilde{H}^T = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} ;$$

we can write Γ_p as

$$\Gamma_p = \begin{bmatrix} \frac{25}{26} & 0 \\ 0 & 0 \end{bmatrix} .$$

Finally, from Eq. (46), we can write R in the form

$$R = \begin{bmatrix} 196 & 55 - i33 \\ 55 + i33 & 33 \end{bmatrix} \begin{cases} \text{total output} \\ \text{power} \end{cases}$$

$$= \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix} \begin{cases} \text{coherent with } \tilde{X} \\ \text{incoherent with } X_1 \end{cases}$$

$$+ \begin{bmatrix} 170 & 55 - i33 \\ 55 + i33 & 29 \end{bmatrix} \begin{cases} \text{incoherent with } \tilde{X} \\ \text{coherent with } X_1 \end{cases}$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \Big\{ \text{noise} .$$

For case II, we will assume that X_c is the only interfering input, so we will remove it from the other two inputs. Thus, we have

$$X_0 = \begin{bmatrix} X_a \\ X_b \end{bmatrix}, \quad X_1 = X_c .$$

We obtain

$$P_0 = \begin{bmatrix} 6 & 5 \\ 5 & 5 \end{bmatrix}, \; P_1 = 3 , \; P_1^{-1} = \tfrac{1}{3} ,$$

$$\tilde{Q} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \; \tilde{H} = \begin{bmatrix} 5 & -i3 \\ 0 & 1 \end{bmatrix}, \; T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

The residual input vector for this case is

$$\tilde{X} = X_0 - TX_1 = \begin{bmatrix} X_a - X_c \\ X_b - X_c \end{bmatrix} = \begin{bmatrix} \xi_1 + \xi_2 \\ \xi_2 \end{bmatrix} .$$

Note that the influence of $X_c = \xi_2$ has been removed from $\tilde{X}$. The residual input power is

$$\tilde{P} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

and

$$\Gamma_i P_0 = P_0 - \tilde{P} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix},$$

so, we get the input coherence matrix

$$\Gamma_i = \begin{bmatrix} 0 & \frac{3}{5} \\ 0 & \frac{3}{5} \end{bmatrix}.$$

Now,

$$\tilde{H}\tilde{P}\tilde{H}^T = \begin{bmatrix} 93 & 10 - i6 \\ 10 + i6 & 2 \end{bmatrix} = \Gamma_p \tilde{R},$$

so

$$\tilde{R} = \tilde{H}\tilde{P}\tilde{H}^T + N = \begin{bmatrix} 94 & 10 - i6 \\ 10 + i6 & 6 \end{bmatrix},$$

from which we can write

$$\Gamma_p = \frac{1}{428} \begin{bmatrix} 422 & 10 - i6 \\ 40 + i24 & 52 \end{bmatrix}.$$

We are now in a position to separate the output power into the following three parts:

$$R = \begin{bmatrix} 196 & 55 - i33 \\ 55 + i33 & 33 \end{bmatrix}$$

$$= \begin{bmatrix} 93 & 10 - i6 \\ 10 + i6 & 2 \end{bmatrix} \begin{cases} \text{coherent with } \tilde{X} \\ \text{incoherent with } X_1 \end{cases}$$

$$+ \begin{bmatrix} 102 & 45 - i27 \\ 45 + i27 & 27 \end{bmatrix} \begin{cases} \text{incoherent with } \tilde{X} \\ \text{coherent with } X_1 \end{cases}$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \{ \text{noise} .$$

Note that the most useful aspect of this theory is the separation of the output-power matrix into parts, in accordance with the partitioning of the input into "desired" and "interfering" groups. The actual element values in the various coherence matrices are difficult to interpret, so it is best to regard coherence matrices as operators on the output power, resulting in the decomposition of this power into several constituents.

V. SUMMARY AND CONCLUSIONS

The coherence concept[1,2] is introduced to describe the division of power at the output of a multiple input–output linear system. Part of this power is directly related (coherent) to the input signal, and the remainder is due to contaminating noise sources. We emphasize the interpretation of each coherence matrix as an operator on the output power, to effect this division between coherent and incoherent contributions.

We discuss the concept of a residual input vector, in which a set of undesired or interfering inputs are removed in a least-squared sense from the remaining desired inputs. In conjunction with this idea, we define an input coherence matrix that characterizes the amount of input power that actually originates with these interfering inputs.

The concept of partial coherence is defined, which characterizes the portion of the output power that actually results from this residual input, in comparison to the incoherent noise power. We show that there are $2^n - 1$ different output coherence matrices associated with an n-input linear system. One of these matrices is the multiple coherence matrix (no interfering inputs), while the remaining $2^n - 2$ matrices are partial-coherence matrices. There are a similar number of input coherence matrices.

There is considerable redundancy in these various matrix representations, since only $(n+m)^2$ real numbers are actually needed to completely describe a linear system with n inputs and m outputs (at any one frequency). Nevertheless, these matrices may be used to present certain information in a form that is relatively easy to interpret from a physical point of view. This is particularly true when interfering-signal contamination must be eliminated.

We close with a numerical example involving a three-input, two-output system, in which the inputs are related. Two cases are illustrated, depending on whether one or two inputs are considered to be interfering.

We have replaced the conventional scalar theory of coherence with a more general matrix approach. We also include the possibility of multiple-interfering signals.

[1] L. D. Enochson, "Frequency Response Functions and Coherence Functions for Multiple Input Linear Systems," NASA Contract No. NAS-5-4590, Washington, D.C., April, 1964 (unpublished).

[2] N. R. Goodman, "Measurement of Matrix Frequency Response Functions and Multiple Coherence Functions," Technical Report No. AFFDL-7R-65-56, Air Force Flight Dynamics Laboratory, Wright–Patterson Air Force Base, OH (unpublished).