Measuring Linear System Parameters
Single Input/Output Transfer & Coherence Functions

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The key to reliable medal parameter estimates is the accurate measurement of the transfer function of a system. In this section, we will develop techniques for estimating transfer and coherence functions of a single input, single output linear system, and we will briefly discuss some of the errors to be expected from these procedures. We will model this linear system as shown below.

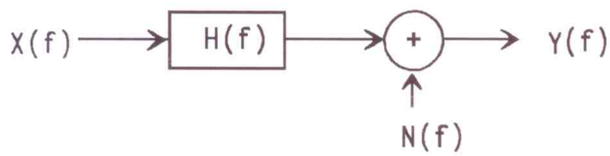

Figure 1. Elementary Linear System Model

The transfer function is $\mathbf{H(f)}$, relating the output $\mathbf{Y(f)}$ to the input $\mathbf{X(f)}$. In addition, we assume some source of contaminating noise $\mathbf{N(f)}$, so we can write the system equation as:

$$\mathbf{Y(f)} = \mathbf{H(f)\,X(f) + N(f)} \qquad (1)$$

This is a frequency domain representation of the system, although we could just as well use a time domain formulation, or a Laplace **s**-plane representation. For brevity, we will generally drop the **f** notation, but this functional dependence is always implied.

In general, we measure **X** and **Y**. and would like to obtain an estimate of **H** from these measurements. In a noise-free environment we could simply divide **Y** by **X** to obtain **H**. However, we usually have substantial amounts of noise, so we must develop another technique.

One of the standard techniques for estimating parameters of noisy signals is the use of a least-squares technique developed by Gauss in 1795 in an effort to improve the orbit determination of minor planets. The procedure is easy to develop, as we show next.

Let's assume that we make n measurements of both **X** and **Y** at each frequency within some band of interest. We want to calculate the value of $\hat{\mathbf{H}}$ at each frequency to minimize the sum of the squared errors between **Y** and $\hat{\mathbf{H}}\mathbf{X}$, where $\hat{\mathbf{H}}$ is an estimate of **H**. We write this squared error as:

$$\varepsilon = \sum_{k=1}^{n} \left|\mathbf{Y} - \hat{\mathbf{H}}\mathbf{X}\right|^2 \qquad (2)$$

We use the absolute magnitude, since these variables are generally quantities. If we write:

$$\varepsilon = \sum_{k=1}^{n} (\mathbf{Y} - \hat{\mathbf{H}}\mathbf{X})\ (\mathbf{Y}^* - \hat{\mathbf{H}}^*\mathbf{X}^*) \qquad (3)$$

we can differentiate ε with respect to $\hat{\mathbf{H}}*$ (or $\hat{\mathbf{H}}$), and set the resulting derivative to zero. Here, we use the * symbol to denote the conjugate of a complex number. Thus, we have

$$\frac{\partial\varepsilon}{\partial\hat{\mathbf{H}}^*} = -\sum_{k=1}^{n} (\mathbf{Y} - \hat{\mathbf{H}}\mathbf{X})\,\mathbf{X}^* = \mathbf{0} \qquad (4)$$

Solving for $\hat{\mathbf{H}}$ gives,

$$\hat{\mathbf{H}} = \frac{\sum_{k=1}^{n} \mathbf{YX}^*}{\sum_{k=1}^{n} \mathbf{XX}^*} \qquad (5)$$

If we imagine $\mathbf{n} - \infty$, we define the cross-power spectrum $\mathbf{G}_{yx}$ and the input auto-power spectrum $\mathbf{G}_{xx}$ by,

$$\mathbf{G}_{yx} = \sum_{k=1}^{\infty} \mathbf{YX}^* \qquad (6)$$

$$\mathbf{G}_{xx} = \sum_{k=1}^{\infty} \mathbf{XX}^* \qquad (7)$$

Then $\hat{\mathbf{H}} = \mathbf{H}$ and we have,

$$\mathbf{H} = \frac{\mathbf{G}_{yx}}{\mathbf{G}_{xx}} \tag{8}$$

In actual practice, **n** is finite, so we define estimates of the cross and auto-spectrum as:

$$\hat{\mathbf{G}}_{yx} = \sum_{k=1}^{n} \mathbf{YX}^* \tag{9}$$

$$\hat{\mathbf{G}}_{xx} = \sum_{k=1}^{n} \mathbf{XX}^* \tag{10}$$

Hence, our least-squares estimate of **H** is,

$$\hat{\mathbf{H}} = \frac{\hat{\mathbf{G}}_{yx}}{\hat{\mathbf{G}}_{xx}} \tag{11}$$

From eq. (1), we have $\mathbf{Y} = \mathbf{HX} + \mathbf{N}$, so

$$\hat{\mathbf{G}}_{yx} = \sum_{k=1}^{n} \mathbf{HXX}^* + \sum_{k=1}^{n} \mathbf{NX}^* = \mathbf{H} \quad \hat{\mathbf{G}}_{xx} + \hat{\mathbf{G}}_{nx} \tag{12}$$

where, $\hat{\mathbf{G}}_{nx}$ is the noise-input cross-spectrum estimate.

$$\hat{\mathbf{G}}_{nx} = \sum_{k=1}^{n} \mathbf{NX}^* \tag{13}$$

Substituting into (11) gives

$$\hat{\mathbf{H}} - \mathbf{H} = \frac{\hat{\mathbf{G}}_{nx}}{\hat{\mathbf{G}}_{xx}} \tag{14}$$

We see that the error in our estimate of **H** is $\frac{\hat{\mathbf{G}}_{nx}}{\hat{\mathbf{G}}_{xx}}$. The noise is unrelated to the input, this approaches zero as $\mathbf{n} \to \infty$. We will discuss this error in more detail later.

Let's substitute this expression for $\hat{\mathbf{H}}$ back into eq. (3), to obtain the squared error for this choice of $\hat{\mathbf{H}}$. We have,

$$\varepsilon = \sum_{k=1}^{n} (\mathbf{YY}^* - \hat{\mathbf{H}}\mathbf{XY}^* - \hat{\mathbf{H}}^*\mathbf{X}^*\mathbf{Y} + \hat{\mathbf{H}}\hat{\mathbf{H}}^*)$$

$$= \hat{\mathbf{G}}_{yy} - \hat{\mathbf{H}}\hat{\mathbf{G}}^*_{yx} - \hat{\mathbf{H}}^*\hat{\mathbf{G}}_{yx} + \left|\hat{\mathbf{H}}\right|^2 \hat{\mathbf{G}}_{xx} \tag{15}$$

From (11) we have, $\hat{\mathbf{H}}\hat{\mathbf{G}}^*_{yx} = \hat{\mathbf{H}}^*\hat{\mathbf{G}}_{yx} = \left|\hat{\mathbf{H}}\right|^2 \hat{\mathbf{G}}_{xx}$, so

$$\varepsilon = \hat{\mathbf{G}}_{yy} - \left|\hat{\mathbf{H}}\right|^2 \hat{\mathbf{G}}_{xx} = \hat{\mathbf{G}}_{yy} - \frac{\left|\hat{\mathbf{G}}_{yx}\right|^2}{\hat{\mathbf{G}}_{xx}} \tag{16}$$

where, $\hat{\mathbf{G}}_{yy}$ is the estimate of the output auto-power spectrum given by

$$\begin{aligned}
\hat{\mathbf{G}}_{yy} &= \sum_{k=1}^{n} \mathbf{YY}^* \\
&= \sum_{k=1}^{n} (\hat{\mathbf{H}}\mathbf{X} + \mathbf{N})\ (\hat{\mathbf{H}}^*\mathbf{X}^* + \mathbf{N}^*) \\
&= \left|\hat{\mathbf{H}}\right|^2 \hat{\mathbf{G}}_{xx} + \hat{\mathbf{H}}\hat{\mathbf{G}}^*_{nx} + \hat{\mathbf{H}}^*\hat{\mathbf{G}}_{nx} + \hat{\mathbf{G}}_{nn} \\
&= \left|\hat{\mathbf{H}}\right|^2 \hat{\mathbf{G}}_{xx} + \hat{\mathbf{G}}_{nn} + (\hat{\mathbf{H}}\hat{\mathbf{G}}^*_{nx} + \hat{\mathbf{H}}^*\hat{\mathbf{G}}_{nx})
\end{aligned} \tag{17}$$

where, $\hat{\mathbf{G}}_{nn}$ is the estimate of the noise auto-power spectrum.

$$\hat{\mathbf{G}}_{nn} = \sum_{k=1}^{n} \mathbf{NN}^* \tag{19}$$

Again, we notice that $\hat{\mathbf{G}}_{nx} \to \mathbf{0}$ as **X** and **N** become incoherent, so

$$\mathbf{G}_{yy} = \left|\mathbf{H}\right|^2 \mathbf{G}_{xx} + \mathbf{G}_{nn} \tag{20}$$

We see that the output power comprises two parts: $\left|\mathbf{H}\right|^2 \mathbf{G}_{xx}$ is directly related to the input, and $\mathbf{G}_{nn}$ is due strictly to the noise. We define the part of $\mathbf{G}_{yy}$ that is coherent with the input as

$$\left|\gamma\right|^2 \mathbf{G}_{yy} = \left|\mathbf{H}\right|^2 \mathbf{G}_{xx} \tag{21}$$

So the incoherent part of $\mathbf{G}_{yy}$ is

$$(1-|\gamma|^2)\ G_{yy} = G_{nn} \tag{22}$$

But, notice from eq. (16) that

$$\varepsilon = (1-|\gamma|^2)\ G_{yy} = G_{nn} \tag{23}$$

Thus, the least squares estimate of **H** produces a squared error equal to the system noise power. Since we assume that this noise is unrelated to the input, there is no way of altering **H** to reduce this squared error any further.

The quantity $|\gamma|^2$ is called the (scalar) coherence function, and describes the division of the output power (G_{yy}) into coherent and incoherent parts (with respect to the input).

Since $\hat{G}_{nx} \neq 0$, we cannot actually measure $|\gamma|^2$, but can only obtain an estimate. Analogous to eq. (21), we define this estimate by

$$|\hat{\gamma}|^2\ \hat{G}_{yy} = |\hat{H}|^2\ \hat{G}_{xx} \text{ , so} \tag{24}$$

$$|\hat{\gamma}|^2 = \frac{|\hat{G}_{yx}|^2}{\hat{G}_{xx}\hat{G}_{yy}} \tag{25}$$

The errors in these estimates of **H** and $|\gamma|^2$ are discussed in references [1] & [2]. For a nonrandom input signal, the probability density of **H** is Gaussian, with zero mean, and variance given by

$$\sigma_H^2 = \frac{|H|^2}{2n}\ \frac{1-|\gamma|^2}{|\gamma|^2} \tag{26}$$

where σ_H^2 is the variance of the real part of **H** as well as the imaginary part of **H**. For a Gaussian random input, uncorreleated with the noise, the probability density of H is a Student's **t**-distribution with zero mean and variance.

$$\sigma_H^2 = \frac{|H|^2}{2(n-1)}\ \frac{1-|\gamma|^2}{|\gamma|^2} \tag{27}$$

Thus, we see that the coherence function can be used to estimate the variance on the estimate of **H**. Note that $\frac{1-|\gamma|^2}{|\gamma|^2}$ is a measure of the output noise power to signal power ratio We should emphasize, however that low coherence values do not necessarily imply poor estimates of **H**. but simply mean that more averaging is needed for a reliable result.

Reference [2] gives a good discussion of the properties of $|\hat{\gamma}|^2$. This is a biased estimator, with maximum bias of $\frac{1}{n}$ for $|\hat{\gamma}|^2 = 0$. The maximum variance is approximately $\frac{8}{27n}$ at $|\gamma|^2 = \frac{1}{3}$. Approximate formulas are,

$$\text{Bias} \approx \frac{1}{n}\ (1-|\nu|^2)^2 \tag{28}$$

$$\text{Variance} \approx \begin{cases} \dfrac{1}{n^2} & \text{for, } |\gamma|^2 = 0 \\[2ex] \dfrac{2}{n}\ |\gamma|^2\ (1-|\gamma|^2)^2, & 0 < |\gamma|^2 \le 1 \end{cases} \tag{29}$$

$$\approx 2|\gamma|^2\ \text{(bias)}$$

The contributions of bias and variance to the error in $|\hat{\gamma}|^2$ are nearly equal for $|\hat{\gamma}|^2$ very small, but the variance is much more dominant, than the bias for most values of $|\hat{\gamma}|^2$.

General Measurement Considerations

There are several factors that contribute to the quality of actual measured transfer and coherence function estimates. Some of the most important sources of error are listed below, along with methods of reducing these errors to a tolerable level, Most of these effects are essentially of nature that limit our ultimate measuring ability,

A) One of the most obvious requirements is to excite the system with energy at all frequencies for which measurements are expected. Be sure that the input signal spectrum does not have "holes" where lit energy exists. Otherwise, the coherence will be very low, and the variance on **H** will be large.

B) We have defined transfer and coherence functions as parameters of a linear system. Non-linearities will generally shirt energy from one frequency to many new frequencies, in away which may be difficult to recognize. The result will be a bias in our estimates of the system parameters, which may not be apparent unless the excitation is changed. One way to reduce the effect of non-linearities is to randomize these contributions by choosing a randomly different input signal for each of the **n** measurements. Subsequent averaging will reduce these contributions, in the same manner that noise is reduced,

C) Some amount of noise is always present, so it is important to average enough measurements together to reduce the variance of our estimates to some level that we can live with. There is always a strong temptation to cut the averaging time to an absolute minimum, and then wonder why the resulting estimates seem to change from one measurement to the next. The effects of variance could easily be interpreted as valid estimates of transfer or coherence function values. Keep in mind that observed values of random variables can easily exceed twice the standard deviation.

D) Any measuring instrument is limited in time resolution, or frequency bandwidth. However, sampling a signal at discrete times also introduces a form of amplitude distortion (called aliasing) that converts high frequency energy to lower frequencies. Thus, the time resolution, and frequency bandwidth parameters, are generally dictated by an analog anti-aliasing filter in front of the sampler. The shape of this filter influences the in-band accuracy and the stop-band rejection characteristics of the instrument. Keep in mind that filters are not perfect, and there is no such thing as a "stop" band, Strong signals always leak through to some extent,

E) Analogous to time resolution limits, there is always a limit on frequency resolution. This is ultimately determined by the total effective time over which coherent data is collected. The effect of this finite collection time is the introduction of another type of non-linear distortion (called leakage), which converts energy at each frequency into energy within a relatively narrow band nearby. This type of distortion is controlled to some extent by weighting (or windowing) the original time domain data. However, there will always be considerable bias in any measurements that are sufficiently close to a strong signal.

There are numerous other sources of potential error, such as overloading the input, extraneous signal pick-up via ground loops or strong electric or magnetic fields nearby, etc. There is no substitute for good estimates of transfer and coherence functions in determining modal parameters from real structures. Considerable care should be exercised in setting up a measurement, and it is very valuable to do a test measurement on a known physical system to verify that all is working well.

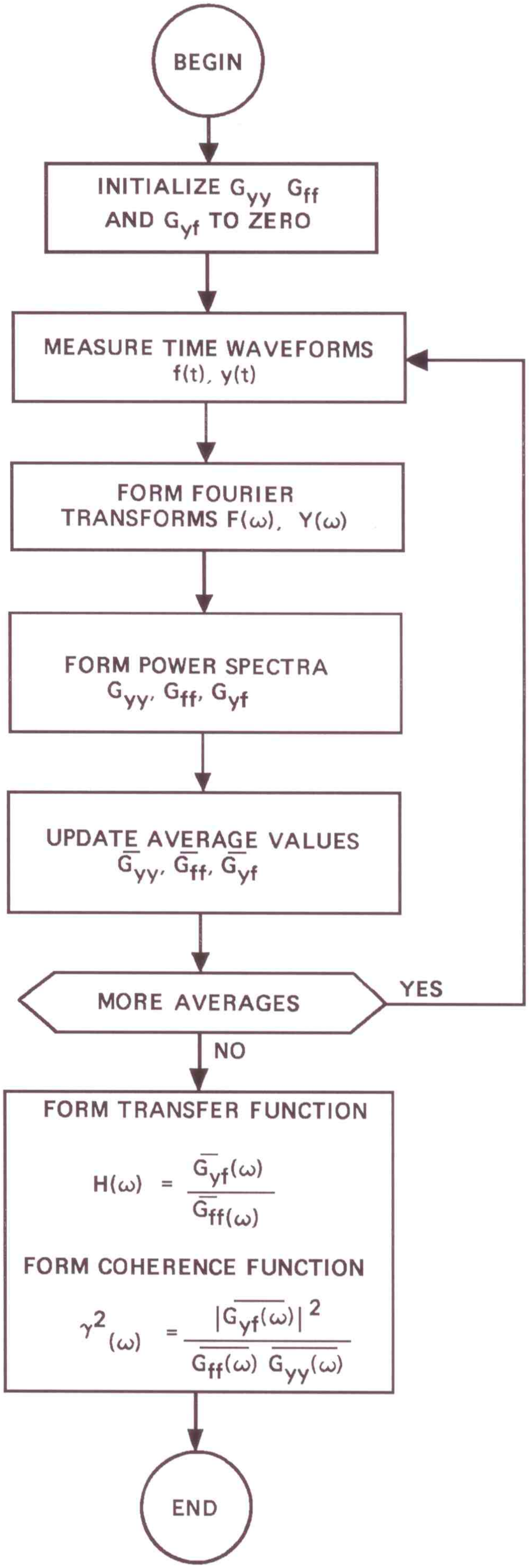

Figure 13. Measurement bar Power Spectrum Averaging'