Proportional Damping from Experimental Data

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ABSTRACT

Damping forces are typically ignored during the Finite Element Analysis (FEA) of mechanical structures. In most real structures, it can be assumed that there are several damping mechanisms at work, but they may be difficult to identify, and even more difficult to model.

Since both mass & stiffness matrices are available during an FEA, a common method of modeling viscous damping is with a *proportional* damping matrix. That is, the viscous damping matrix is assumed to be a *linear combination* of the mass & stiffness matrices. Therefore, in order to model viscous damping with a *proportional* damping matrix, the two constants of proportionality must be determined.

In this paper, a least-squared-error relationship between *experimental* modal frequency & damping and the proportional damping *constants of proportionality* is developed. An example is included in which experimental modal parameters are used to calculate the constants of proportionality. The modal parameters of an FEA model with proportional damping are then compared with the original experimental modal parameters.

KEY WORDS

Viscous Damping
Proportional Damping
Modal Damping
Finite Element Analysis
Experimental Modal Parameters

INTRODUCTION

All experimental resonant vibration data is characterized by a *decaying sinusoidal response* when all forces are removed from the structure. The overall response is modeled as a *summation* of contributions, each term due to a mode of vibration. Each modal contribution is itself a *decaying sinusoidal* function. The decay envelope for each mode is modeled with a decreasing exponential function, and the decay constant in the exponent is called the modal *damping coefficient*. It is also called the *half power point*, or *3dB point* damping.

VISCOUS DAMPING

It is commonly assumed that displacement of the surrounding air by the surfaces of a vibrating structure is a *dominant* damping mechanism at work in most structures, at least those in earth's atmosphere. It is also assumed that this mechanism can be adequately modeled using a linear viscous damping model.

A linear viscous damping model, in which the damping or dissipative forces are proportional to the surface velocity, is used as the time domain model for this type of damping.

The time domain linear differential equations of motion for a vibrating structure with viscous damping are written as;

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K]\{x(t)\}=\{f(t)\} \tag{1}$$

where:

$[M]$ = mass matrix (n by n)
$[C]$ = viscous damping matrix (n by n)
$[K]$ = stiffness matrix (n by n)
$\{\ddot{x}(t)\}$ = accelerations (n-vector)
$\{\dot{x}(t)\}$ = velocities (n-vector)
$\{x(t)\}$ = displacements (n-vector)
$\{f(t)\}$ = external forces (n-vector)
n = number of degrees-of-freedom of the model

Equation (1) is a *force balance* between the *internal* (*inertial*, *dissipative*, and *restoring*) forces on the left-hand side and the *external*ly *applied* forces on the right-hand side.

Equation (1) describes the *linear*, *stationary*, *viscously damped*, dynamic behavior of a structure. The mass [M] & stiffness [K] matrices are typically synthesized from the physical properties and geometry of the structure, using an FEA software program. However, in most FEA practice today, the viscous damping matrix [C] is assumed to be zero. That is, damping is ignored altogether.

The frequency domain version of this equation is commonly used as the basis for determining the modes of a structure. Modes are solutions to the *homogeneous* form of this equation, shown as equation (2) below;

$$\left([M]p^2+[C]p+[K]\right)\{\phi\}=\{0\}$$
$$p=-\sigma+j\omega \tag{2}$$

Each *non-trivial* solution of this matrix equation consists of a pole location, p (also called an *eigenvalue*) and a mode shape, $\{\phi\}$ (also called an *eigenvector*). Each complex pole is made up of both the *damping decay constant* (σ) and the *damped natural frequency* (ω).

PROPORTIONAL DAMPING MATRIX

A proportional damping matrix is assumed to be a *linear combination* of the mass & stiffness matrices. That is, the viscous damping forces are *assumed* to be proportional to the inertial and restoring forces in the structure.

$$[C] = \alpha[M] + \beta[K] \tag{3}$$

α = constant of mass proportionality
β = constant of stiffness proportionality

Once α & β have been determined, all of the matrices in equation (2) are known, and the modes of the damped structure can be calculated. The question now becomes; ***"How can α & β be determined for a real structure?"***

PROPORTIONAL DAMPING COEFFCIENTS

Modal frequency & damping estimates are routinely determined from experimental data using modern modal testing and analysis methods. Experimental forced vibration data is commonly obtained in the form of a set of Frequency Response Functions (FRFs). An FRF is a special form of a Transfer Function. Its numerator is the Fourier spectrum of a structural *output* (acceleration, velocity, or displacement response), and its denominator is the Fourier spectrum of the *input* (the force that caused the response).

Modal frequency & damping estimates are obtained from one or more FRFs by *curve fitting* them, using an analytical model that includes frequency & damping as unknown parameters. A set of modal frequency & damping estimates can therefore be obtained for all modes in the frequency band of the FRF measurements.

These experimental frequency & damping estimates can then be used to calculate the proportional damping matrix coefficients, α & β. The relationship between modal frequency & damping and α & β is derived from equation (2).

Substituting equation (3) into equation (2) and re-arranging terms gives;

$$\left([M]p^2 + (\alpha[M]+\beta[K])p + [K]\right)\{\phi\} = \{0\}$$
$$\left([M](p^2 + \alpha p) + [K](\beta p + 1)\right)\{\phi\} = \{0\}$$
$$\left([M]\left((-\sigma + j\omega)^2 + \alpha(-\sigma + j\omega)\right) + [K](\beta(-\sigma + j\omega) + 1)\right)\{\phi\} = \{0\}$$
$$\left([M]\left((\sigma^2 - \omega^2 - j2\sigma\omega) + (-\alpha\sigma + j\alpha\omega)\right) + [K](\beta(-\sigma + j\omega) + 1)\right)\{\phi\} = \{0\}$$
$$\left([M]\left((\sigma^2 - \omega^2 - \alpha\sigma) + j(-2\sigma\omega + \alpha\omega)\right) + [K]\left((-\sigma\beta + 1) + j\beta\omega\right)\right)\{\phi\} = \{0\} \tag{4}$$

If the damping term were removed from Equation (2), it would be the homogeneous equation of motion for an *un-damped* structure. Notice that equation (4) also has the same form as an equation for an *un-damped* structure. A known property of the mode shapes $\{\phi\}$ of an *un-damped* structure is that they are *real-valued*. Modes with *real-valued* mode shapes are also called *normal* modes.

Because $\{\phi\}$ is real-valued, the real and imaginary parts of equation (4) are not coupled. Therefore, the *real* and *imaginary* parts of equation (4) can be written as separate equations;

$$\begin{aligned}&\left([M](\sigma^2 - \omega^2 - \alpha\sigma) + [K](-\sigma\beta + 1)\right)\{\phi\} = \{0\}\\&\left([M](-2\sigma\omega + \alpha\omega) + [K]\beta\omega\right)\{\phi\} = \{0\}\end{aligned} \tag{5}$$

Putting these equations into the standard form for an *un-damped* structure;

$$\begin{aligned}&\left(\frac{(\sigma^2 - \omega^2 - \alpha\sigma)}{(-\sigma\beta + 1)}[M] + [K]\right)\{\phi\} = \{0\}\\&\left(\frac{(-2\sigma + \alpha)}{\beta}[M] + [K]\right)\{\phi\} = \{0\}\end{aligned} \tag{6}$$

Both of these equations must be satisfied for a proportionally damped structure. The eigensolution to these equations has *unique* poles (or eigenvalues), and the coefficients of the mass matrix can be equated to each of the poles. The equation for one of the poles of the *un-damped* structure is;

$$\frac{(\sigma^2 - \omega^2 - \alpha\sigma)}{(-\sigma\beta + 1)} = \frac{(-2\sigma + \alpha)}{\beta} = -\Omega^2 \tag{7}$$

where;
$\Omega^2 = (\sigma^2 + \omega^2)$ = an *un-damped* natural frequency *squared.*

This gives us a single equation with 2 unknowns (α & β) in it;

$$2\sigma = \alpha + \beta\Omega^2 \tag{8}$$

or

$$2\sigma = \alpha + \beta(\sigma^2 + \omega^2) \tag{9}$$

Equation (9) can be used together with estimates of frequency & damping for *two or more* modes to compute the proportional damping constants, α & β.

LEAST-SQUARED-ERROR SOLUTION

Given a set of modal frequencies & damping for **n modes (n>=2)**, then **n equations** can be written;

$$2\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_n\end{Bmatrix}=\begin{bmatrix}1 & \Omega_1^2\\ 1 & \Omega_2^2\\ \vdots & \vdots\\ 1 & \Omega_n^2\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \tag{10}$$

This is an *over-specified* set of linear equations. The least-squared-error solution of these equations is written as;

$$2\begin{bmatrix}1 & 1 & \cdots & 1\\ \Omega_1^2 & \Omega_2^2 & \cdots & \Omega_n^2\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_n\end{Bmatrix}=\begin{bmatrix}1 & 1 & \cdots & 1\\ \Omega_1^2 & \Omega_2^2 & \cdots & \Omega_n^2\end{bmatrix}\begin{bmatrix}1 & \Omega_1^2\\ 1 & \Omega_2^2\\ \vdots & \vdots\\ 1 & \Omega_n^2\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \tag{11}$$

or;

$$2\begin{Bmatrix}\sum_{i=1}^{n}\sigma_i\\ \sum_{i=1}^{n}\sigma_i\Omega_i^2\end{Bmatrix}=\begin{bmatrix}N & \sum_{i=1}^{n}\Omega_i^2\\ \sum_{i=1}^{n}\Omega_i^2 & \sum_{i=1}^{n}\left(\Omega_i^2\right)^2\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \tag{12}$$

Equation (12) can be solved for α & β with the following matrix equation;

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=2\begin{bmatrix}N & \sum_{i=1}^{n}\Omega_i^2\\ \sum_{i=1}^{n}\Omega_i^2 & \sum_{i=1}^{n}\left(\Omega_i^2\right)^2\end{bmatrix}^{-1}\begin{Bmatrix}\sum_{i=1}^{n}\sigma_i\\ \sum_{i=1}^{n}\sigma_i\Omega_i^2\end{Bmatrix} \tag{13}$$

Equation (13) is the desired relationship between the frequency & damping estimates for multiple modes and the proportional damping matrix coefficients, α & β.

Least-squared-error estimates of α & β can be calculated using any number (2 or greater) of experimental modal frequencies & damping and equation (13). These estimates can then be used to add a proportional damping matrix to an FEA model.

If the *damped* FEA model is then solved for its modes, the following question arises; *"How well do the modal frequencies & damping of the damped FEA model match the experimental modal parameters from which the damped model was derived?"* This question is addressed with the following examples.

BEAM STRUCTURE

We will consider the modes of the beam structure shown in Figure 1, also called the Jim Beam. This beam consists of three aluminum plates fastened together with cap screws. The top plate is fastened to the back plate with three screws, and the bottom plate is also fastened to the back plate with three screws.

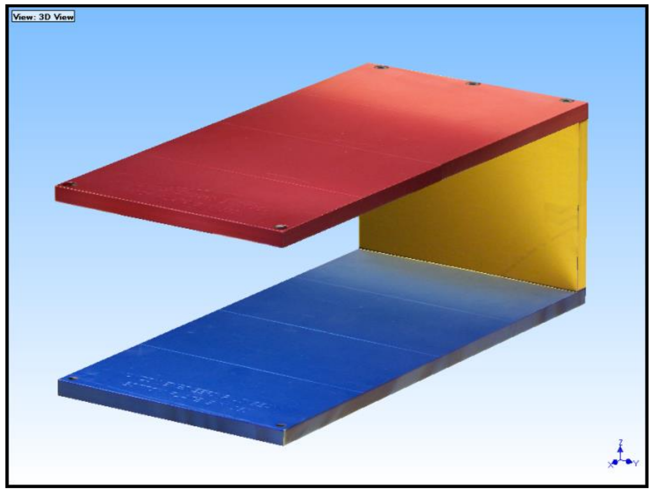

Figure 1 Jim Beam Structure.

Modal Frequency & Damping

The modal frequency & damping for the first 11 (lowest frequency) *un-damped* FEA modes and the Experimental Modal Analysis (EMA) modes of the Jim Beam are listed in Table 1. The FEA frequencies were obtained as the eigenvalues of an *un-damped* FEA model, [1]. Because the FEA model was *un-damped*, the FEA modes have no damping. The EMA parameters were estimated by *curve fitting* a set of experimentally derived FRFs.

Mode	FEA Freq. (Hz)	FEA Damp. (Hz)	EMA Freq. (Hz)	EMA Damp. (Hz)	Mode Shape MAC
1	61.405	0.0	96.944	5.6347	0.74
2	143.81	0.0	164.95	3.1125	0.96
3	203.71	0.0	224.57	6.5223	0.96
4	310.62	0.0	347.56	5.1552	0.95
5	414.4	0.0	460.59	11.502	0.93
6	442.6	0.0	492.82	4.6424	0.96
7	583.44	0.0	635.18	14.247	0.94
8	1002.2	0.0	1108.2	4.964	0.90
9	1090.8	0.0	1210.5	7.1292	0.88
10	1168.3	0.0	1322.6	7.2498	0.84
11	1388.2	0.0	1555.1	17.112	0.84

Table 1. Un-Damped FEA & Damped EMA Modes.

Each FEA modal frequency is less than its corresponding EMA frequency, indicating that the FEA model was *less stiff* than the actual Jim Beam structure. However, the Modal Assurance Criterion (MAC) values between the mode shape pairs indicate that the FEA and EMA mode shapes are comparable. Two mode shapes are strongly correlated if their MAC value is *0.90 or greater*.

A modal model consisting of 11 modes is called a *truncated* modal model because *all* of the modes of the actual Jim Beam are not present in the model. In principle, all real structures have an *infinite* number of modes, but because all FEA models contain a *finite* number of DOFs, they will only yield a finite number of modes, or a *truncated* modal model. Similarly, because FRF measurements are made over a *finite* frequency band, all experimentally derived modal models are *truncated* models.

TWO EXTREME CASES

Two extreme cases are possible with the coefficients α & β, namely, $\alpha > 0$, $\beta = 0$ and $\alpha = 0$, $\beta > 0$.

Case #1 (β = 0)

If $\beta = 0$, then viscous damping is *only proportional* to the mass distribution, and equation (8) reduces to;

$$\alpha = 2\sigma \qquad (14)$$

Assume that the Jim Beam is proportionally damped, and that its proportional damping matrix coefficients are; $\alpha = 2\pi$, $\beta = 0$. Then equation (14) says that *all* modes of the beam will have the *same* modal damping;

$$\sigma = \pi \text{ rad/sec} = 0.5 \text{ Hz}.$$

The coefficients $\alpha = 2\pi$, $\beta = 0$ were used to create a proportional damping matrix from the mass & stiffness matrices of the *un-damped* FEA model for the Jim Beam, and the *damped* FEA model was solved for its modes. The expected result (all modes with damping = 0.5 Hz), is shown in Table 2.

Mode	Un-damped FEA Freq. (Hz)	Un-damped FEA Damp. (Hz)	Damped FEA Freq. (Hz)	Damped FEA Damp. (Hz)	Mode Shape MAC
1	61.405	0.0	61.403	0.49975	1.00
2	143.81	0.0	143.81	0.49975	1.00
3	203.71	0.0	203.71	0.49975	1.00
4	310.62	0.0	310.62	0.49975	1.00
5	414.4	0.0	414.4	0.49975	1.00
6	442.6	0.0	442.6	0.49975	1.00
7	583.44	0.0	583.44	0.49975	1.00
8	1002.2	0.0	1002.2	0.49975	1.00
9	1090.8	0.0	1090.8	0.49975	1.00
10	1168.3	0.0	1168.3	0.49975	1.00
11	1388.2	0.0	1388.2	0.49975	1.00

Table 2. Proportionally Damped FEA Modes ($\alpha = 2\pi$, $\beta = 0$***).***

Case #2 (α = 0)

If $\alpha = 0$, then viscous damping is *only proportional* to the stiffness distribution, and equation (8) reduces to;

$$\beta\Omega = \frac{2\sigma}{\Omega} = 2\zeta \qquad (15)$$

where $\zeta = \frac{\sigma}{\Omega}$ is the *percent of critical* damping of a mode.

Assume that the FEA model of the Jim Beam is proportionally damped, and $\alpha = 0$. Equation (15) says that for a given value of β, the *percent of critical* damping of each mode is proportional to its *un-damped* frequency. For $\zeta = 1\%$ for the first mode (with frequency 61.4 Hz), equation (15) gives **β=0.0000518**.

The coefficients **α = 0, β=0.0000518** were used to create a proportional damping matrix from the mass & stiffness matrices of the *un-damped* FEA model, and the *damped* FEA model was solved for its modes. The results are shown in Table 3. Notice that the 61.4 Hz mode has the expected 1 % damping, and also that the *percent of critical* damping *increases* as the modal frequency of the other modes increases.

Mode	Un-damped FEA Freq. (Hz)	Un-damped FEA Damp. (Hz)	Damped FEA Freq. (Hz)	Damped FEA Damp. (Hz)	Damped FEA Damp. (%)
1	61.405	0.0	61.402	0.61361	0.99928
2	143.81	0.0	143.77	3.3657	2.3403
3	203.71	0.0	203.6	6.7533	3.3151
4	310.62	0.0	310.23	15.702	5.0549
5	414.4	0.0	413.46	27.946	6.7437
6	442.6	0.0	441.45	31.879	7.2026
7	583.44	0.0	580.81	55.396	9.4947
8	1002.2	0.0	988.81	163.46	16.31
9	1090.8	0.0	1073.5	193.63	17.751
10	1168.3	0.0	1147	222.11	19.012
11	1388.2	0.0	1352.3	313.61	22.591

Table 3. Damped FEA Modes (α = 0, β = 0.0000518).

The list of EMA damping values in Table 1 shows a wide range of values, from 3.11 Hz to 17.11 Hz. Clearly, neither of the two extreme proportional damping cases exists in the Jim Beam; $\beta = 0$ which gives modes with the *same* damping, or $\alpha = 0$ which give modes with *percent of critical* damping that *increases* with increasing frequency.

Nevertheless, all of the EMA frequency & damping estimates in Table 1 can be used in equation (13) to calculate least-squared-error estimates of α & β. These estimates can in turn be used to create a proportionally *damped* FEA model for the Jim Beam.

USING EMA FREQUENCY & DAMPING

The EMA frequency & damping for the 11 modes in Table 1 were used to calculate α & β for the Jim Beam using equation (13). The least-squared-error estimates of α & β were;

$$\alpha = 76.6972, \beta = 8.0835\text{ e-7}$$

These coefficients were used to create a proportional damping matrix from the mass & stiffness matrices of the *un-damped* FEA model. This model was solved for its modes, and the modal frequency & damping of the *damped* FEA model of the Jim Beam are shown in Table 4.

The modal damping values of the *damped* FEA model do exhibit monotonically increasing values with frequency, indicating their stronger proportionality to the stiffness matrix than to the mass matrix. This is similar to extreme Case #2. Even though the FEA damping values don't *closely match* the EMA damping values, they are *in the range* of the EMA values. This is a desirable property for making the FEA model useful for modeling the dynamics of the real structure.

Mode	Damped FEA Freq. (Hz)	Damped FEA Damp. (Hz)	EMA Freq. (Hz)	EMA Damp. (Hz)	Mode Shape MAC
1	61.1	6.1129	96.944	5.6347	0.73
2	143.68	6.1559	164.95	3.1125	0.97
3	203.62	6.2088	224.57	6.5223	0.96
4	310.56	6.3484	347.56	5.1552	0.96
5	414.35	6.5395	460.59	11.502	0.93
6	442.55	6.6008	492.82	4.6424	0.96
7	583.4	6.9678	635.18	14.247	0.94
8	1002.2	8.6542	1108.2	4.964	0.92
9	1090.8	9.125	1210.5	7.1292	0.90
10	1168.2	9.5695	1322.6	7.2498	0.86
11	1388.2	10.997	1555.1	17.112	0.84

Table 4. Damped FEA Modes (α = 76.6972, β = 8.0835 e-7).

USING FEA FREQUENCY & EMA DAMPING

Each FEA frequency in Table 1 is *less than* the frequency of its corresponding EMA frequency, indicating that the FEA model is *less stiff* than the real Jim Beam, as tested. To determine the influence of modal frequency on the calculation of α & β values, the FEA frequencies were used instead of the EMA frequencies in equation (13). For this case, the least-squared-error estimates of α & β were;

$$\alpha = 76.4183, \beta = 1.0202\text{ e-6}$$

These estimates were then used to create a proportional damping matrix from the mass & stiffness matrices of the *un-damped* FEA model. The modal parameters of the *damped* FEA model are again compared with the EMA parameters of the Jim Beam in Table 5.

Mode	Damped FEA Freq. (Hz)	Damped FEA Damp. (Hz)	EMA Freq. (Hz)	EMA Damp. (Hz)	Mode Shape MAC
1	61.102	6.0933	96.944	5.6347	0.73
2	143.68	6.1475	164.95	3.1125	0.97
3	203.62	6.2142	224.57	6.5223	0.96
4	310.56	6.3904	347.56	5.1552	0.96
5	414.35	6.6316	460.59	11.502	0.93
6	442.55	6.7091	492.82	4.6424	0.96
7	583.4	7.1723	635.18	14.247	0.94
8	1002.2	9.3007	1108.2	4.964	0.92
9	1090.8	9.8949	1210.5	7.1292	0.90
10	1168.2	10.456	1322.6	7.2498	0.86
11	1388.2	12.258	1555.1	17.112	0.84

Table 5. Damped FEA Modes (α = 76.4183, β = 1.0202 e-6).

The modal damping values of the *damped* FEA model are monotonically increasing with frequency, again indicating proportionality more like extreme Case #2. These FEA damping values are *closer* to the EMA values than when the EMA frequencies are used, but the differences between the two solutions are *not significant*. The mode shape MAC values for this case are identical to the case where the EMA frequencies were used to estimate α & β.

CONCLUSIONS

An equation was derived for calculating estimates of the proportional damping matrix coefficients α & β from experimental modal frequency & damping. The equation for obtaining the α & β estimates was derived as a least-squared-error solution to an over-specified set of linear equations, where α & β are functions of modal frequency & damping alone.

These coefficients can be used to create a *damped* FEA model for a structure, which is more clearly useful for simulation studies than an *un-damped* model. This approach provides a straightforward way to add a viscous damping matrix to any FEA model, and solve for its FEA modes which include realistic values of modal damping.

The damping values of a two differently *damped* FEA models were compared with the EMA damping estimates for the same structure. Although the FEA modal damping didn't match well on a mode-by-mode basis with all of the EMA estimates, the damping values of matching mode pairs were *similar in value*.

The strongest reason for this disparity in FEA versus EMA values is that the proportional damping matrix is *restricted* by the distribution of mass & stiffness in the FEA model. On the other hand, the damping *mechanisms* that influenced the experimental damping values were most likely *distributed differently* than the mass & stiffness.

In fact, the Jim Beam structure was tested while resting on a foam rubber pad, which clearly would have greater damping influence on the lower plate than on the other two plates. Therefore, *depending on their mode shape deformations* on the lower plate, some modes would be more strongly influenced than others by the damping provided from the foam base. This alone could account for the wide range of modal damping values (3.11 Hz to 17.11 Hz) among the EMA modes. Because it was modeled using the mass & stiffness of the FEA model, the modal damping of the damped FEA model not only increased monotonically with frequency, but also had less range of values (6.09 Hz to 12.2 Hz).

From our example, it can be concluded that a proportionally damped FEA model doesn't necessarily yield modes with damping that perfectly match experimental damping values. Nevertheless, this approach is a useful and straightforward way to add *realistic viscous damping* to a *un-damped* FEA model. A *damped* FEA model, using proportional viscous damping derived from experimental data, is a lot closer to modeling the dynamics of a real structure than an *un-damped* FEA model.

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