Linear Superposition and Modal Participation

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ABSTRACT
Modes of vibration are defined as solutions to a set of linear differential equations which characterize the resonant dynamic behavior of structures. One of the properties of these linear equation solutions is ***superposition***. That is, the overall response of a structure can be represented as a ***summation*** of the responses of each of the modes.

In this paper, it is shown how the superposition property of mode shapes can be used to;

- **Represent** Operating Deflection Shapes (ODS's) as a ***summation*** of mode shape contributions.
- **Expand** a set of shapes using a set of mode shapes with more DOFs in them.
- **Decompose** a set of frequency or time domain waveforms into a ***summation*** of resonance curves.
- **Scale** a set of EMA mode shapes, OMA mode shapes or ODS's using a modal model (a set of scaled mode shapes).
- Derive the **Modal Assurance Criterion (MAC)** as a measure of the correlation between pairs of shapes.

All of these applications lend more meaning to the term ***modal participation***, which is commonly used to characterize structural vibration as a summation of resonant contributions. This new definition of ***modal participation*** is illustrated with several examples.

KEY WORDS
Operating Deflection Shapes (ODS's)
Experimental Modal Analysis (EMA) Mode Shapes
Finite Element Analysis (FEA) Mode Shapes
Modal Assurance Criterion (MAC)

INTRODUCTION
When all excitation forces are removed from a structure, its resonant vibration response is characterized by a time domain ***decaying sinusoidal waveform***, as shown in Figure 1. This resonant response can be modeled as a ***summation of contributions*** due to each of the structure's resonances. Each resonant contribution is itself modeled with a ***decaying sinusoidal*** waveform. The frequency of each resonant response is invariant unless the physical properties or boundary conditions of the structure change. Each resonant frequency is therefore called a ***natural frequency*** of the structure.

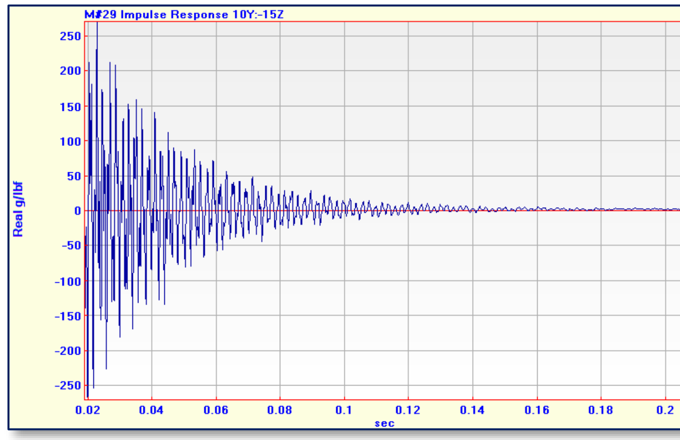


Figure 1. Resonant Response after Forces Removed

The decay envelope of each resonant response is modeled with a decreasing exponential function, and the coefficient in the exponential term is called the ***damping decay constant***.

A mode of vibration is a ***compact mathematical description*** of a structural resonance. Not only are modal parameters solutions to a set of differential equations, but they are also used to model the resonant vibration of a real structure, assuming that it behaves in a linear dynamic manner.

The natural frequency of each structural resonance is also called its ***modal frequency***. Likewise, the damping decay constant of a resonance is also called its ***modal damping coefficient***. This damping is also called the ***half power point*** or ***3dB point*** damping [5].

BACKGROUND
In this paper two different kinds of shapes will be discussed; Operating Deflection Shapes (ODS's) and Mode Shapes.

ODS: The response at a frequency or time value, of ***two or more*** DOFs on a structure. A DOF is motion at a point in a direction. [7].

Therefore, the values at each frequency sample of a set of frequency domain functions (Auto & Cross spectra, Fourier spectra, FRFs, etc.) ***is an ODS***.

Likewise, the values at each sample of a set of time domain functions (Sinusoidal Responses, Impulse Response Functions, Auto & Cross Correlations, etc) ***is also an ODS***.

Mode Shape: A mode shape can either be an experimental (EMA) mode shape, derived by curve fitting a set of experimentally derived FRFs, or it can be an analytical (FEA) mode shape, an eigenvector calculated as part of an ***eigensolution*** to a set of linear homogeneous differential equations that model the dynamics of a structure [8].

Equating Two Sets of Shapes

Two sets of complex valued shapes can be assembled into two matrices **([U] & [V])**, where ***each column*** of each matrix contains a shape. The shapes in matrix **[U]** can be thought of as ***un-scaled*** shapes, and the shapes in matrix **[V]** as ***scaled*** shapes. The two shape matrices can be equated to each other with the following matrix equation,

$$[\mathrm{U}][\mathrm{W}] = [\mathrm{V}] \qquad (1)$$

$[\mathrm{V}]$ = matrix of **scaled** complex shapes
$[\mathrm{U}]$ = matrix of **un-scaled** complex shapes
$[\mathrm{W}]$ = matrix of **complex scale factors**

Writing out the matrices in terms of their components,

$$\begin{bmatrix} u_{1,1} & \cdots & u_{1,n_u} \\ \vdots & \ddots & \vdots \\ u_{m,1} & \cdots & u_{m,n_u} \end{bmatrix} \begin{bmatrix} w_{1,1} & \cdots & w_{1,n_s} \\ \vdots & \ddots & \vdots \\ w_{n_u,1} & \cdots & w_{n_u,n_s} \end{bmatrix} = \begin{bmatrix} v_{1,1} & \cdots & v_{1,n_s} \\ \vdots & \ddots & \vdots \\ v_{m,1} & \cdots & v_{m,n_s} \end{bmatrix}$$

(m by n_u) (n_u by n_s) (m by n_s)

n_u = number of **un-scaled** shapes
n_s = number of **scaled** shapes
m = number of **matching shape DOFs** or shape components

Least-Squared-Error Solution

Equation (2) below is the least-squared-error solution of Equation (1). In addition to requiring a matrix inverse, the other requirement for a solution is that the two matrices [U] & [V] have ***at least some matching shape DOFs***, or shape components.

$$[\mathrm{W}] = \left[[\mathrm{U}]^{\mathrm{h}}[\mathrm{U}]\right]^{-1}[\mathrm{U}]^{\mathrm{h}}[\mathrm{V}] \qquad (2)$$

h - denotes the transposed conjugate matrix
-1 - denotes the inverse matrix

MODAL PARTICIPATION FACTORS

When [U] is a matrix of mode shapes, and [V] is a matrix of ODS's, then ***each column*** of the scale factor matrix **[W]** is a measure of how much each mode shape ***contributes to*** or ***participates*** in each ODS. If [W] is a diagonal matrix (with ***non-zero diagonals*** and zeros everywhere else), then each ODS is being ***dominated*** by a ***single*** mode shape.

Example #1: Mode Shapes Dominating ODS's

For this example, ODS's of the Jim Beam structure shown in Figure 2 were obtained by ***saving the cursor values*** at each resonance peak in the ***imaginary part*** of a set of experimental FRFs.

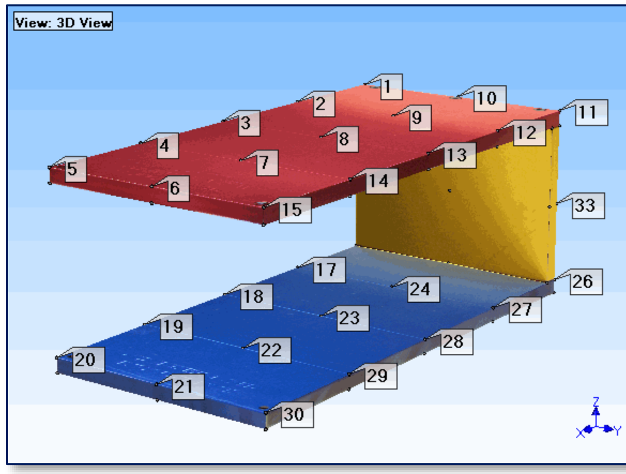


Figure 2 Jim Beam Structure.

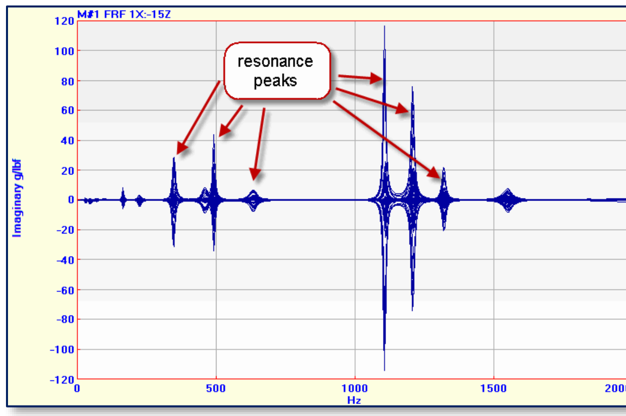


Figure 3. Imaginary Parts of FRFs Overlaid.

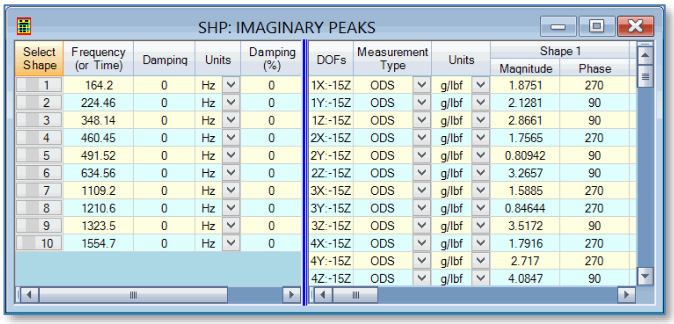
Figure 3 shows the imaginary parts of the FRFs overlaid on one another.

The FRFs were acquired from an impact test of the Jim Beam structure. The beam was impacted at point 15 in the vertical (Z) direction, and a tri-axial accelerometer was roved to the 33 different (numbered) points on the beam. An FRF was calculated between the impact force applied at DOF (15Z) and each of the resulting acceleration responses (3 DOFs at each Point). The ODS's taken as the resonance peak values of the 99 FRFs are listed in Figure 4.

An FEA model of the Jim Beam structure was also built and solved for its mode shapes, or ***eigenvectors*** [3]. The FEA mode shapes are listed in Figure 5.

The Jim Beam model was meshed to provide more DOFs before solving for its modes. Hence, the FEA mode shapes had 630 (translational and rotational) DOFs in them, but only 99 matched with the DOFs of the ODS's.

Using these two shape matrices, the scale factor matrix [W] was calculated using equation (2). Its ***magnitudes*** are shown in Figure 6.

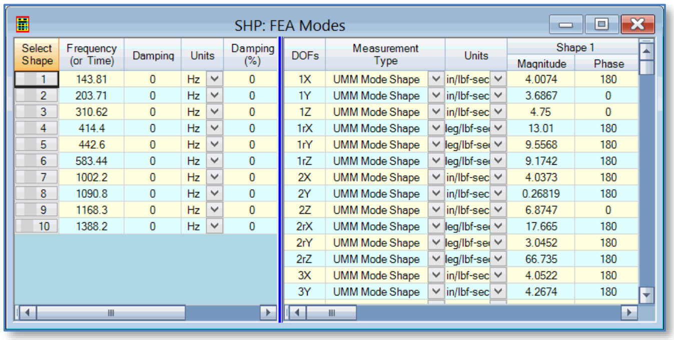


SHP: IMAGINARY PEAKS

Select Shape	Frequency (or Time)	Damping	Units	Damping (%)
1	164.2	0	Hz	0
2	224.46	0	Hz	0
3	348.14	0	Hz	0
4	460.45	0	Hz	0
5	491.52	0	Hz	0
6	634.56	0	Hz	0
7	1109.2	0	Hz	0
8	1210.6	0	Hz	0
9	1323.5	0	Hz	0
10	1554.7	0	Hz	0

DOFs	Measurement Type	Units	Shape 1 Magnitude	Shape 1 Phase
1X:-15Z	ODS	g/lbf	1.8751	270
1Y:-15Z	ODS	g/lbf	2.1281	90
1Z:-15Z	ODS	g/lbf	2.8661	90
2X:-15Z	ODS	g/lbf	1.7565	270
2Y:-15Z	ODS	g/lbf	0.80942	90
2Z:-15Z	ODS	g/lbf	3.2657	90
3X:-15Z	ODS	g/lbf	1.5885	270
3Y:-15Z	ODS	g/lbf	0.84644	270
3Z:-15Z	ODS	g/lbf	3.5172	90
4X:-15Z	ODS	g/lbf	1.7916	270
4Y:-15Z	ODS	g/lbf	2.717	270
4Z:-15Z	ODS	g/lbf	4.0847	90

Figure 4. Jim Beam ODS's.



SHP: FEA Modes

Select Shape	Frequency (or Time)	Damping	Units	Damping (%)
1	143.81	0	Hz	0
2	203.71	0	Hz	0
3	310.62	0	Hz	0
4	414.4	0	Hz	0
5	442.6	0	Hz	0
6	583.44	0	Hz	0
7	1002.2	0	Hz	0
8	1090.8	0	Hz	0
9	1168.3	0	Hz	0
10	1388.2	0	Hz	0

DOFs	Measurement Type	Units	Shape 1 Magnitude	Shape 1 Phase
1X	UMM Mode Shape	in/lbf-sec	4.0074	180
1Y	UMM Mode Shape	in/lbf-sec	3.6867	0
1Z	UMM Mode Shape	in/lbf-sec	4.75	0
1rX	UMM Mode Shape	leg/lbf-se	13.01	180
1rY	UMM Mode Shape	leg/lbf-se	9.5568	180
1rZ	UMM Mode Shape	leg/lbf-se	9.1742	180
2X	UMM Mode Shape	in/lbf-sec	4.0373	180
2Y	UMM Mode Shape	in/lbf-sec	0.26819	180
2Z	UMM Mode Shape	in/lbf-sec	6.8747	0
2rX	UMM Mode Shape	leg/lbf-se	17.665	180
2rY	UMM Mode Shape	leg/lbf-se	3.0452	180
2rZ	UMM Mode Shape	leg/lbf-se	66.735	180
3X	UMM Mode Shape	in/lbf-sec	4.0522	180
3Y	UMM Mode Shape	in/lbf-sec	4.2674	180

Figure 5. Jim Beam FEA Shapes.

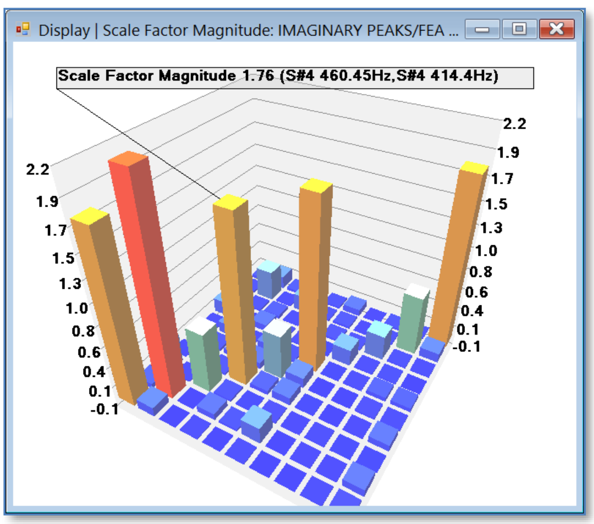


Figure 6. Participation of FEA Shapes in Jim Beam ODS's.

This matrix is ***nearly diagonal***, indicating that each experimentally derived ODS is being ***dominated*** by an FEA mode shape. In other words, each FEA mode shape is a ***close representation*** of the spatial amplitude distribution of each structural resonance.

SHAPE EXPANSION

After a participation matrix [W] has been calculated between the matching DOFs of two shape matrices ([U] & [V]), an expanded set of shapes can be calculated with the following equation,

$$\begin{bmatrix} V_m \\ V_u \end{bmatrix} = \begin{bmatrix} U_m \\ U_u \end{bmatrix} [W] \qquad (3)$$

$[U_m], [V_m]$ = sub-matrices of <u>matching</u> shape DOFs

$[U_u], [V_u]$ = sub-matrices of <u>un-matched</u> shape DOFs

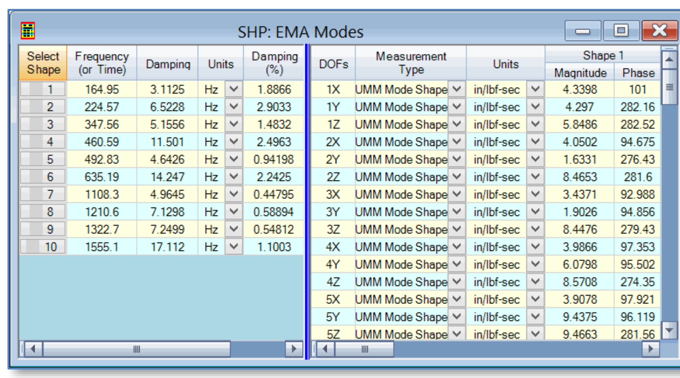
Equation (3) is useful for;

1. Expanding an ODS with a few DOFs in it using a set of mode shapes with more DOFs in them.
2. Expanding a set of EMA mode shapes using a set of FEA mode shapes with more DOFs in them.

Example #2: Expanding EMA Mode Shapes

In some experimental situations, it may not be possible to measure the structural responses at all desired DOFs because some DOFs are inaccessible. However, if a set of FEA mode shapes ***correlate well*** with the EMA mode shapes at matching DOFs (meaning that the [W] is a ***nearly diagonal*** matrix), then equation (3) can be used to expand the EMA mode shapes to include the ***un-measured*** DOFs.

Figure 7 shows the EMA mode shapes of the Jim Beam. These modal parameters were obtained by curve fitting the 99 experimental FRFs acquired during an impact test of the test article. Figure 8 is a bar chart of the scaling matrix [W] between the EMA & FEA mode shapes. Because it is ***nearly diagonal*** it indicates that the FEA mode shapes ***correlate one for one*** with the EMA mode shapes at the 99 matching DOFs.



SHP: EMA Modes

Select Shape	Frequency (or Time)	Damping	Units	Damping (%)
1	164.95	3.1125	Hz	1.8866
2	224.57	6.5228	Hz	2.9033
3	347.56	5.1556	Hz	1.4832
4	460.59	11.501	Hz	2.4963
5	492.83	4.6426	Hz	0.94198
6	635.19	14.247	Hz	2.2425
7	1108.3	4.9645	Hz	0.44795
8	1210.6	7.1298	Hz	0.58894
9	1322.7	7.2499	Hz	0.54812
10	1555.1	17.112	Hz	1.1003

DOFs	Measurement Type	Units	Shape 1 Magnitude	Shape 1 Phase
1X	UMM Mode Shape	in/lbf-sec	4.3398	101
1Y	UMM Mode Shape	in/lbf-sec	4.297	282.16
1Z	UMM Mode Shape	in/lbf-sec	5.8486	282.52
2X	UMM Mode Shape	in/lbf-sec	4.0502	94.675
2Y	UMM Mode Shape	in/lbf-sec	1.6331	276.43
2Z	UMM Mode Shape	in/lbf-sec	8.4653	281.6
3X	UMM Mode Shape	in/lbf-sec	3.4371	92.988
3Y	UMM Mode Shape	in/lbf-sec	1.9026	94.856
3Z	UMM Mode Shape	in/lbf-sec	8.4476	279.43
4X	UMM Mode Shape	in/lbf-sec	3.9866	97.353
4Y	UMM Mode Shape	in/lbf-sec	6.0798	95.502
4Z	UMM Mode Shape	in/lbf-sec	8.5708	274.35
5X	UMM Mode Shape	in/lbf-sec	3.9078	97.921
5Y	UMM Mode Shape	in/lbf-sec	9.4375	96.119
5Z	UMM Mode Shape	in/lbf-sec	9.4663	281.56

Figure 7. EMA Modes of the Jim Beam.

Equation (3) was then used to solve for the expanded EMA shapes, and the MAC values between the expanded EMA shapes and the FEA shapes are shown in Figure 9. (MAC is discussed in a succeeding section of this paper. [1], [2], [4]) The MAC values in Figure 9 show that each expanded EMA shape also correlates well (is ***co-linear*** with) an FEA shape.

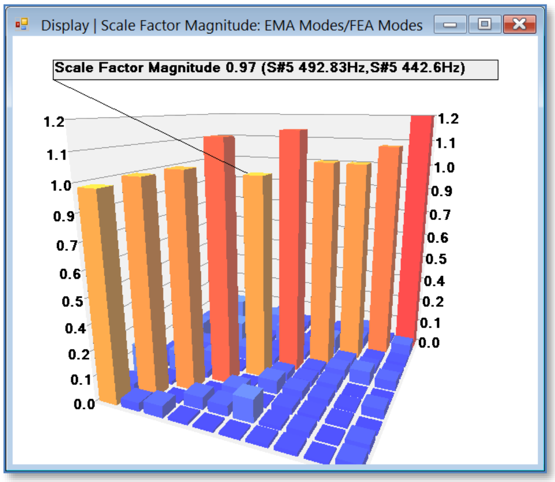


Figure 8. Scaling Matrix for EMA & FEA Mode Shapes.

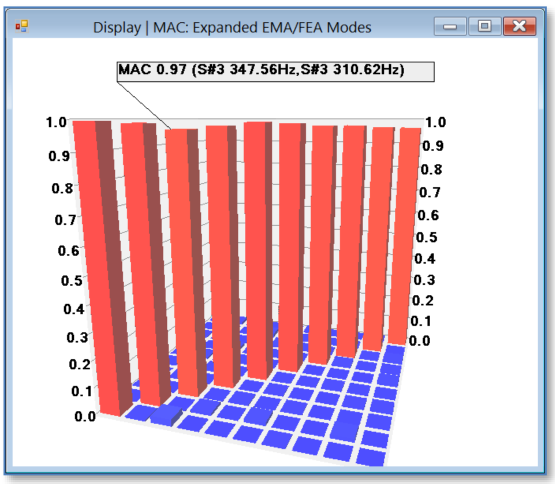


Figure 9. MAC values Between Expanded EMA & FEA Shapes.

MODAL DECOMPOSITION OF WAVEFORMS

A set of mode shapes with DOFs that match the DOFs in a set of time or frequency functions can be used to decompose those waveforms into contributions from each of the modes. The scale factor matrix [W] in equation (1) will therefore contain a column of scale factors corresponding to each time or frequency sample. These columns result in a set of waveforms showing how each mode participates in the overall structural response at each time or frequency value.

Example #3: Modal Decomposition of FRFs

In this example the EMA mode shapes of the Jim Beam will be used to decompose the experimental FRFs into multiple resonance waveforms, one for each mode. Equation (2) was used to calculate the decomposition (scale factors) of the FRFs at each frequency.

An FRF is overlaid together with their ten modal resonance curves in Figure 10. Notice how each resonance curve ***dominates*** the overall FRF response by having a peak ***at or near*** each resonance peak in the FRF.

It is important to note that this decomposition ***only requires the mode shapes***. Modal frequency & damping are not used. In fact, the resonance curves ***can be curve fit*** to obtain modal frequency & damping estimates.

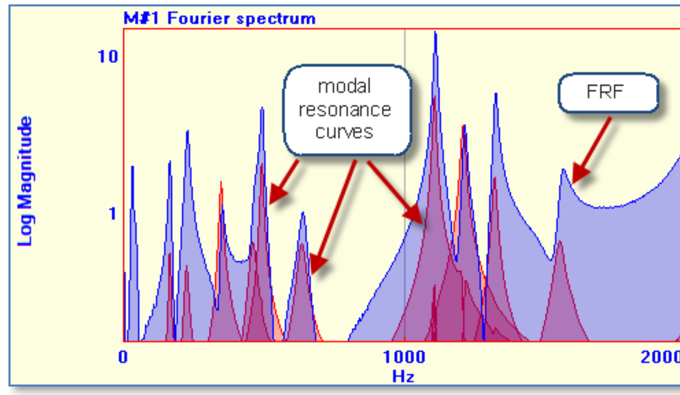


Figure 10. FRFs Decomposed Using EMA Mode Shapes.

MODAL MODEL

Mode shapes are eigenvectors, and as such are only unique in ***"shape"***, not in value. Therefore, mode shapes don't normally have units associated with them.

However, if a set of mode shapes is scaled to properly account for the mass & stiffness properties of a structure, it is called a ***modal model***. Modal models ***do have units*** associated with them. A modal model is useful for several modeling & simulation applications;

- **FRF Synthesis**
  - FRFs can be created between ***any two DOFs*** of the mode shapes.
  - Overlaid synthesized & experimental FRFs provide a graphical comparison.
  - FRAC values between synthesized & experimental FRFs provide a numerical comparison.
- **MIMO Modeling**
  - Time or frequency waveforms can be used.
  - Multiple Outputs calculated from Multiple Inputs.
  - Multiple Inputs calculated from Multiple Outputs.
- **Structural Dynamics Modification (SDM)**
  - Provides rapid investigation of many ***"what if?"*** structural modifications.
  - Modes of the ***unmodified*** structure plus ***modification elements*** attached to a geometric model are required.
  - FEA elements are used by solution is ***much faster*** than FEA eigensolutions.

One of the popular ways to create a ***modal model*** is to scale the mode shapes to yield **Unit Modal Masses**. FEA mode shapes are commonly scaled using **Unit Modal Mass** scaling [3]. Scaled mode shapes have units associated with them which are **(response units/(force units - seconds))**.

Example #4: Scaling Mode Shapes to Unit Modal Masses

In this example, the EMA mode shapes of the Jim Beam will be scaled using its FEA mode shapes, which are already scaled to Unit Modal Masses. First, equation (2) is solved for the scale factors [W], where [U] = the un-scaled EMA mode shapes and [V] = the FEA shapes. The magnitudes of the [W] are shown in Figure 8. Next, the scale factors [W] and the un-scaled EMA mode shapes [U] are used in equation (3) to scale the EMA mode shapes to **Unit Modal Masses**.

To confirm the scaling, the scaled EMA mode shapes and the FEA mode shapes are again used in equation (2) to calculate scale factors, which are shown in Figure 10. The diagonal scale factors are ***nearly all "1"***, indicated that the EMA mode shapes and FEA mode shapes are both scaled to Unit Modal Masses.

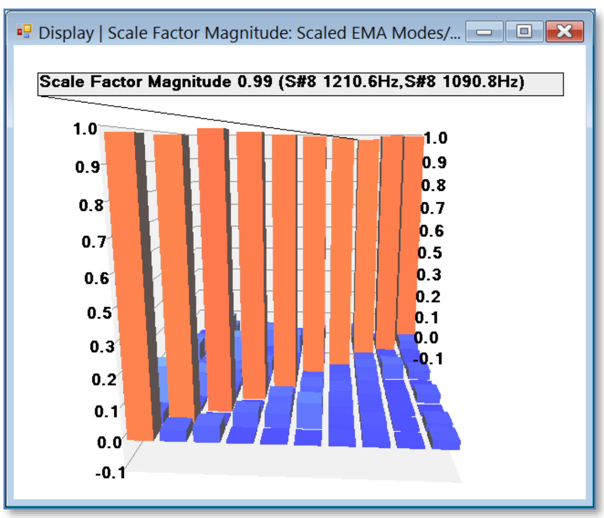


Figure 10. Scaling Matrix After UMM Scaling of EMA Shapes.

MODAL ASSURANCE CRITERION (MAC)

For two shapes $\{u_i\}$ and $\{v_i\}$, the scaling equation reduces to,

$$\begin{Bmatrix} u_{1,i} \\ \vdots \\ u_{m,i} \end{Bmatrix} [w_{i,j}] = \begin{Bmatrix} v_{1,j} \\ \vdots \\ v_{m,j} \end{Bmatrix} \qquad (4)$$

m = number of **matching shape DOFs** or shape components

The single scale factor for equating the two shapes is,

$$w_{i,j} = \frac{\{u_{1,i}^* \ \cdots \ u_{m,i}^*\}\begin{Bmatrix} v_{1,j} \\ \vdots \\ v_{m,j} \end{Bmatrix}}{\{u_{1,i}^* \ \cdots \ u_{m,i}^*\}\begin{Bmatrix} u_{1,i} \\ \vdots \\ u_{m,i} \end{Bmatrix}} \qquad (5)$$

* - denotes the **complex conjugate**

Now, if $\{u_i\}$ and $\{v_i\}$ are interchanged in equation (4), the scale factor that equates them is written,

$$z_{i,j} = \frac{\{v_{1,i}^* \ \cdots \ v_{m,i}^*\}\begin{Bmatrix} u_{1,j} \\ \vdots \\ u_{m,j} \end{Bmatrix}}{\{v_{1,i}^* \ \cdots \ v_{m,i}^*\}\begin{Bmatrix} v_{1,i} \\ \vdots \\ v_{m,i} \end{Bmatrix}} \qquad (6)$$

The Modal Assurance Criterion is simply the product of the two scale factors,

$$MAC_{i,j} = w_{i,j} z_{i,j}$$

$$= \frac{\left| \{u_{1,i}^* \ \cdots \ u_{m,i}^*\}\begin{Bmatrix} v_{1,j} \\ \vdots \\ v_{m,j} \end{Bmatrix} \right|^2}{\{u_{1,i}^* \ \cdots \ u_{m,i}^*\}\begin{Bmatrix} u_{1,i} \\ \vdots \\ u_{m,i} \end{Bmatrix}\{v_{1,j}^* \ \cdots \ v_{m,j}^*\}\begin{Bmatrix} v_{1,j} \\ \vdots \\ v_{m,j} \end{Bmatrix}}$$

MAC has values ***between 0 & 1***. For $\{u_i\} = \{v_j\}$ **MAC = 1**. Otherwise, **MAC < 1**.

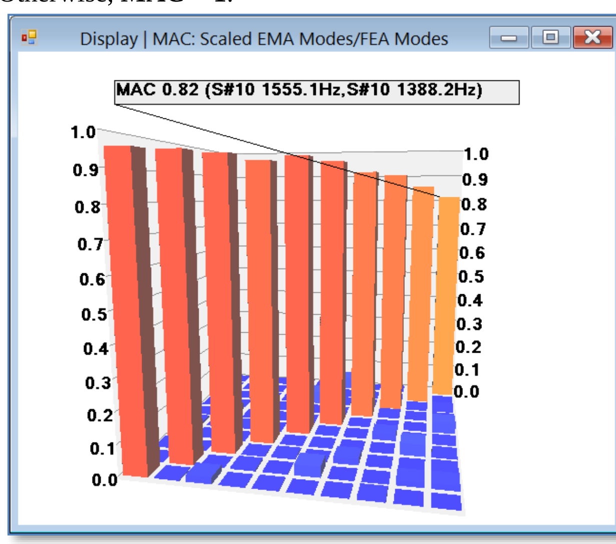


Figure 11 MAC values between EMA & FEA mode shapes.

CONCLUSIONS

Several applications of the linear relationship between two shape matrices were explored in this paper. Each shape matrix can consist of ODS's, EMA mode shapes, OMA mode shapes, FEA mode shapes, or any matrix, the columns of which can be called ***"shapes"***. The linear relationship in Equation (1) ***only involves the shapes themselves***, not their frequencies (or damping in the case of EMA mode shapes), and is only valid for ***matching DOFs*** or shape components between the two matrices.

It was shown that this relationship can be used for scaling shapes, expanding shapes, and for time or frequency waveform decomposition. It was also shown how the Modal Assurance Criterion (MAC) is derived from this relationship [1], [2].

It was also pointed out that when the scaled matrix [V] contains ODS's and the un-scaled matrix [U] contains mode shapes, the columns of the scale factor matrix [W] are a measure of the ***participation*** of each mode in each ODS, more commonly known as ***modal participation factors***. Moreover, when a set of time or frequency domain waveforms is decomposed using mode shapes, the decomposition at each time or frequency sample a measure of the modal participation at that (time or frequency) sample.

All of these examples verify that the linear superposition property of mode shapes is useful in a number of different ways for visualizing and understanding how the resonant vibration of a mechanical structure can be characterized in terms of its modes.

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