

**COMPILATION OF TIME WINDOWS
AND LINE SHAPES FOR
FOURIER ANALYSIS**

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INTRODUCTION

The following pages comprise a compilation of a dozen time windows and line shapes (along with their most significant parameters) for use in digital Fourier analysis. These functions have been selected to cover most measurement situations of practical interest, although an infinite number of variations are possible.

There exists a general theory of line shapes which result from finite time windows that can be summarized as follows:

If a discrete time window of finite duration produces N samples of time data spaced Δt apart, then the Fourier transform of this time window (line shape) will have exactly $N-1$ zeros in the (complex) frequency plane in a region bounded by $\frac{1}{\Delta t} (m - 1/2) < \text{Re}(f) \leq \frac{1}{\Delta t} (m + 1/2)$, for each integer m .

Thus, all possible line shapes can be obtained simply by moving zeros around in the complex frequency plane. Generally, zeros are removed from the main lobe and either 1.) placed at "infinity" to increase the side lobe roll-off rate, or 2.) placed among the near side lobes to reduce side lobe amplitude, or 3.) placed along the imaginary frequency axis to flatten the main lobe top.

This is a rectangular window along with its associated sinc function line shape. This shape occurs by default when no other window is used. The main lobe is narrow, but side lobes are very large and roll off slowly. The main lobe top is quite rounded and can introduce large measurement errors.

This window is only useful when an integer number of periods of the time waveform are included in the window. Otherwise, leakage is very severe.

DEFINITION

Line Shape:
$$L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}}, \quad N = 1024$$

Time Window:
$$w(x) = \Pi(x) = 1 \text{ for } |x| < .5$$

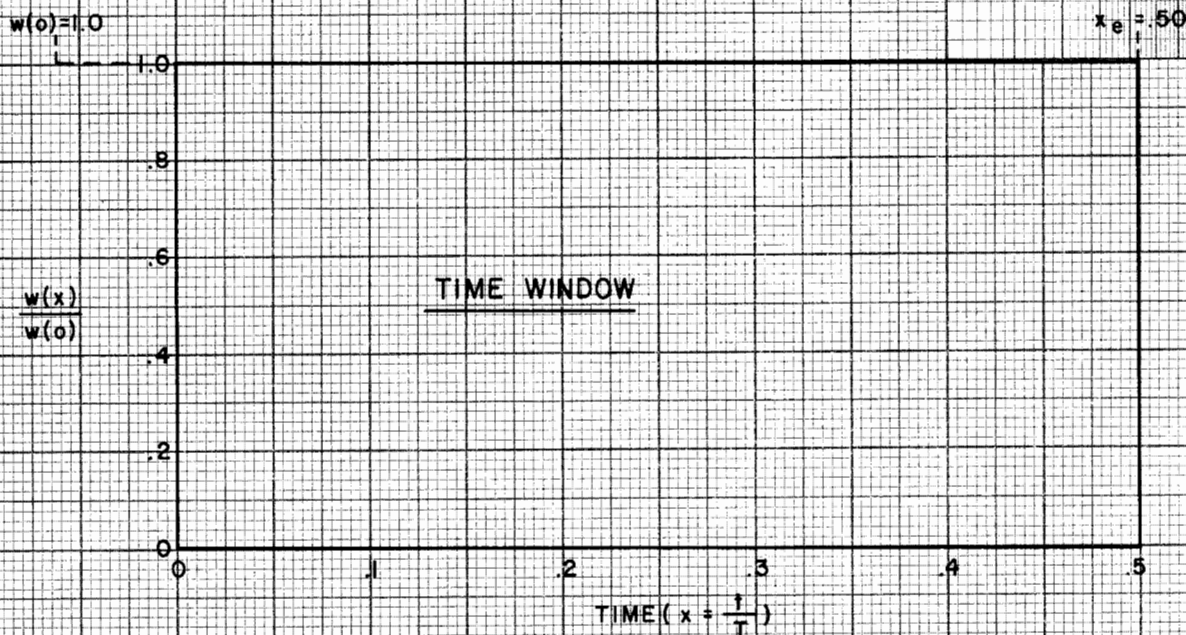
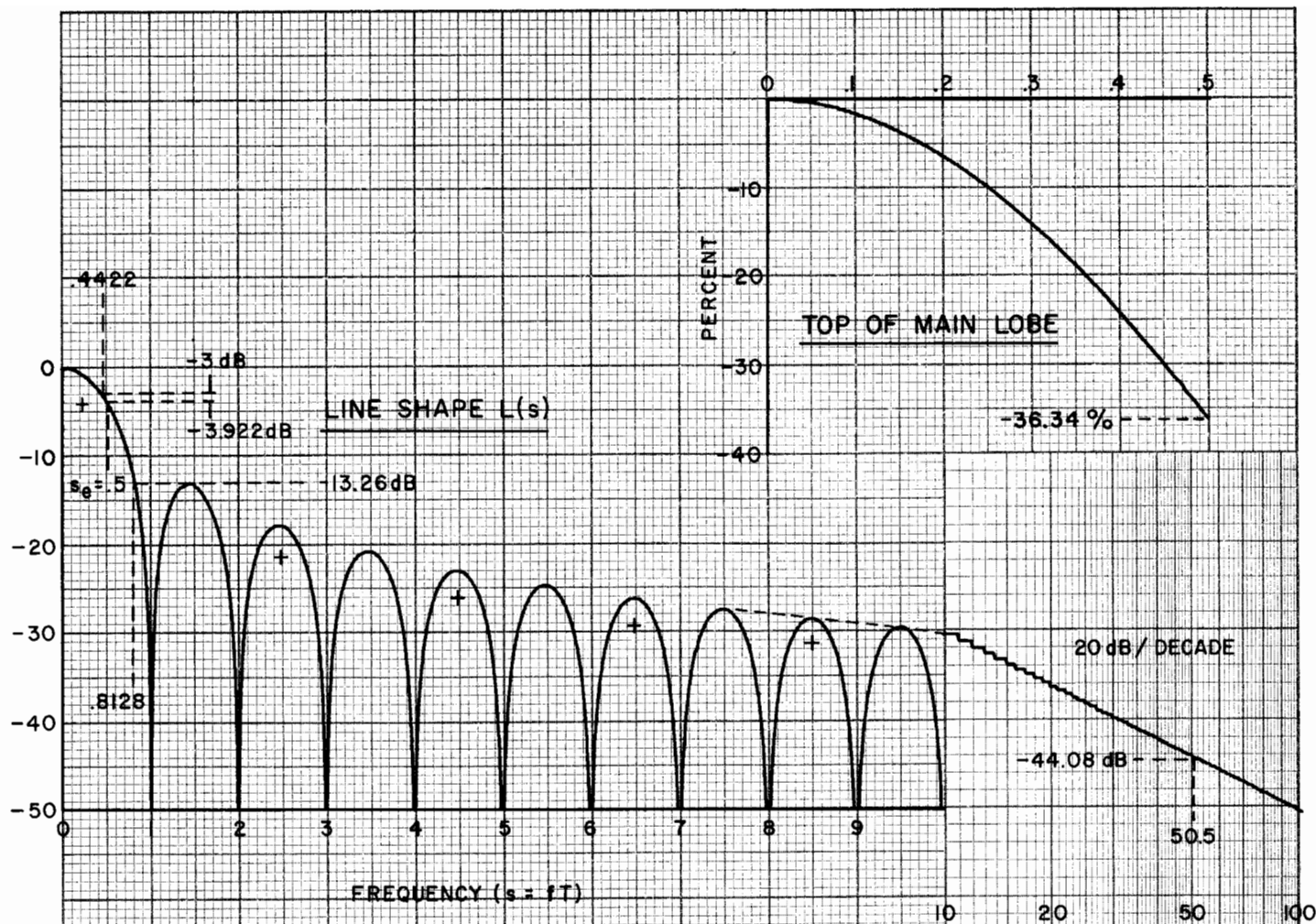
$$= 0 \text{ for } |x| > .5$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	1.0
	(Line Shape) ²	1.0
	Side Lobes	3.661
	(Side Lobes) ²	.1884

Variance Reduction:	50% Time Record Overlap	.75
	90% Time Record Overlap	.6576
	Quadratic Frequency Smoothing	1.0

DECIBELS



This is commonly called a Hamming window. It is obtained by moving a main lobe zero from $s = 1$ to $s = 2.6491$, thereby splitting the first side lobe into two parts. The main lobe is still relatively narrow, and the sidelobes are reasonably small. However, the side lobes roll off slowly.

This window is useful for resolving closely spaced frequencies where amplitude accuracy (18%) is not important.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-1}^1 a_k \delta(s-k), \quad a_{-k} = a_k, \quad N = 1024$$

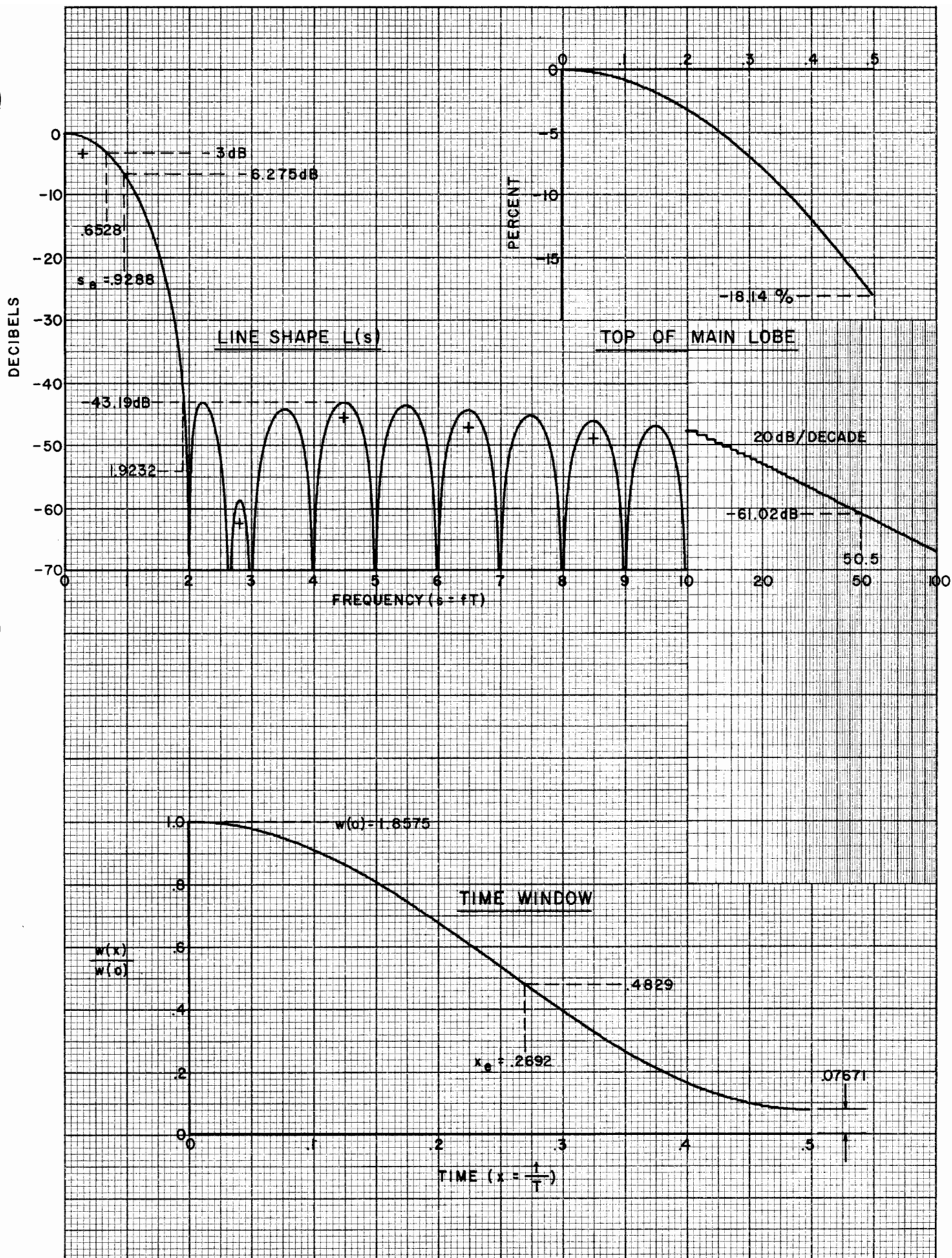
$$\text{Time Window: } w(x) = \Pi(x) \cdot [a_0 + 2a_1 \cos 2\pi x]$$

$$\text{Where: } a_0 = 1.0 \quad a_1 = .428752$$

$$\text{Polynomial Zeros: } s = \pm 2.6491$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	1.8575
	(Line Shape) ²	1.3677
	Side Lobes	.3990
	(Side Lobes) ²	8.602 x 10 ⁻⁴
Variance Reduction:	50% Time Record Overlap	.5534
	90% Time Record Overlap	.4656
	Quadratic Frequency Smoothing	.5708



This is commonly called the Hanning window, and enjoys considerable popularity because it is very easy to implement. This window is available as a standard keyboard command on the HP Fourier Analyzer. The first few side lobes are rather large, but the 60 dB/decade roll off rate is very helpful. This line shape is constructed by moving the main lobe zero at $s = 1$ to "infinity".

This window is most useful for searching operations where good frequency resolution is needed, but amplitude accuracy (15%) is not important. It is particularly useful where a large dynamic range between spectral lines occurs.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-1}^1 a_k \delta(s-k), \quad a_{-k} = a_k, \quad N = 1024$$

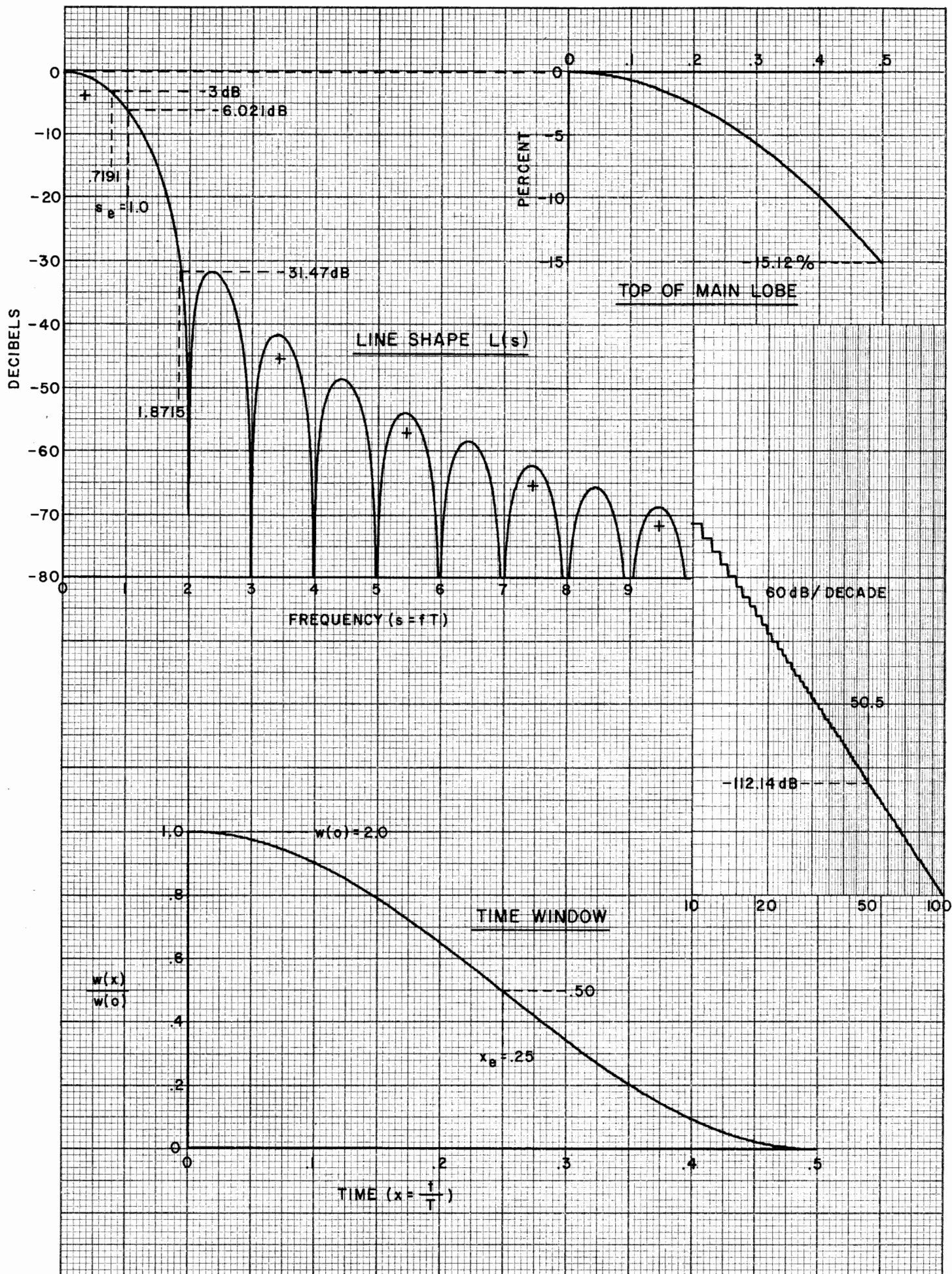
$$\text{Time Window: } w(x) = \Pi(x) \cdot [a_0 + 2a_1 \cos 2\pi x]$$

$$\text{Where: } a_0 = 1.0 \quad a_1 = .50$$

There are no polynomial zeros.

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	2.0
	(Line Shape) ²	1.5
	Side Lobes	8.485×10^{-2}
	(Side Lobes) ²	1.346×10^{-3}
Variance Reduction:	50% Time Record Overlap	.5278
	90% Time Record Overlap	.4307
	Quadratic Frequency Smoothing	.50



By allowing some main lobe broadening, it is possible to reduce the side lobe amplitude to a very respectable level. This line shape has side lobes that are 71.48 dB below the main lobe. However, the roll off rate is slow.

This shape is very useful in situations where small spectral lines must be resolved in the presence of large lines. However, amplitude accuracy is poor (12%).

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-2}^2 a_k \delta(s-k), a_{-k} = a_k, N = 1024$$

$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^2 a_k \cos 2\pi kx \right]$$

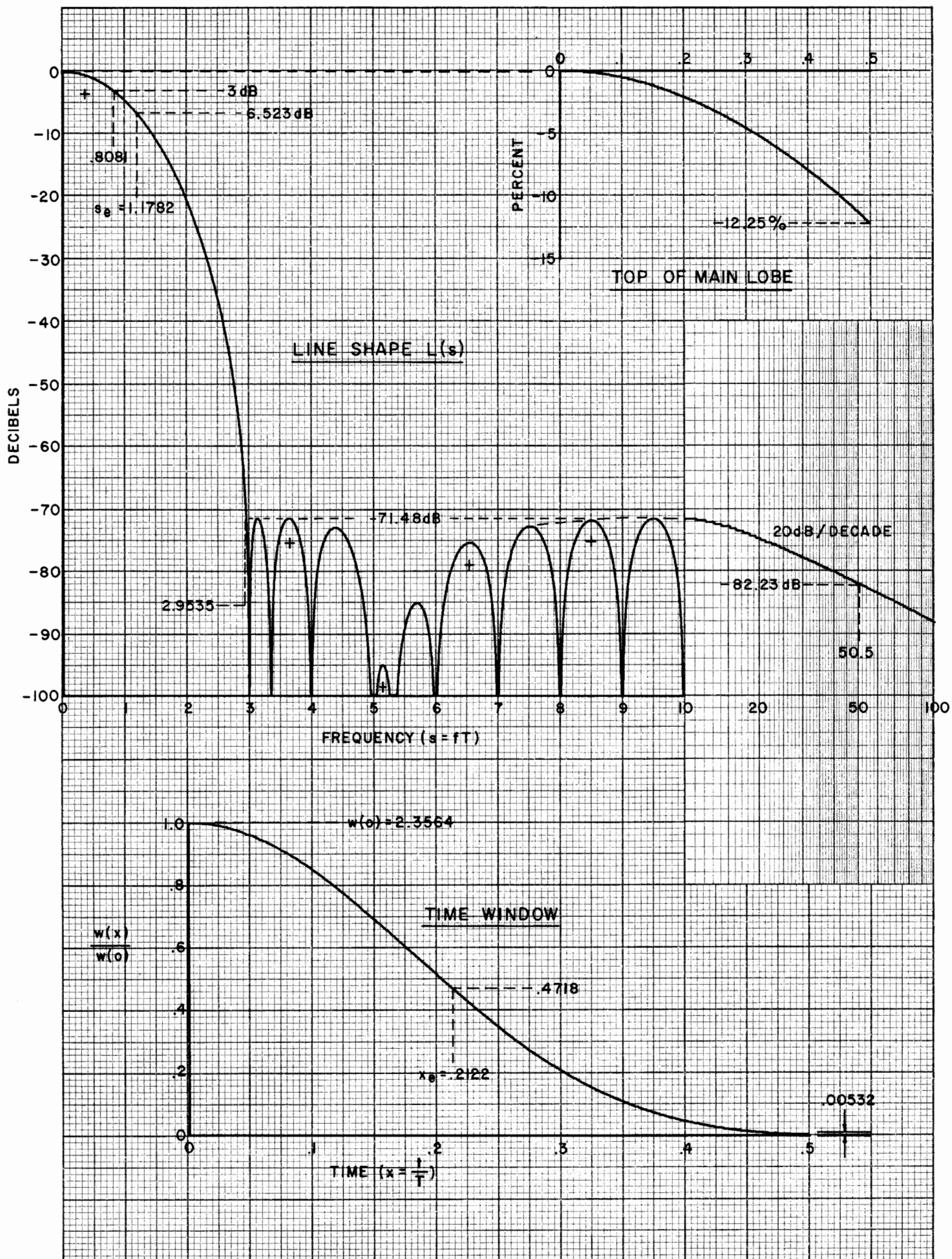
$$\text{Where: } a_0 = 1 \qquad a_2 = .0922245$$

$$a_1 = .585956$$

$$\text{Polynomial Zeros: } s = \pm 3.35247, \pm 5.3282$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	2.3564
	(Line Shape) ²	1.7037
	Side Lobes	2.919 x 10 ⁻²
	(Side Lobes) ²	2.997 x 10 ⁻⁶
Variance Reduction:	50% Time Record Overlap	.5094
	90% Time Record Overlap	.3882
	Quadratic Frequency Smoothing	.4258



One of the zeros is placed at $s = i1.412364$ to flatten the main lobe top. Thus, a single spectral line can be measured with an amplitude accuracy of $\pm .1\%$. The side lobes are relatively small, but roll off slowly.

This line shape is most useful when reasonable amplitude accuracy is needed ($\pm 1\%$), and only a small number of spectral lines are present. The dynamic range is rather limited, but frequency resolution is relatively good.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-2}^2 a_k \delta(s-k), a_{-k} = a_k, N = 1024$$

$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^2 a_k \cos 2\pi kx \right]$$

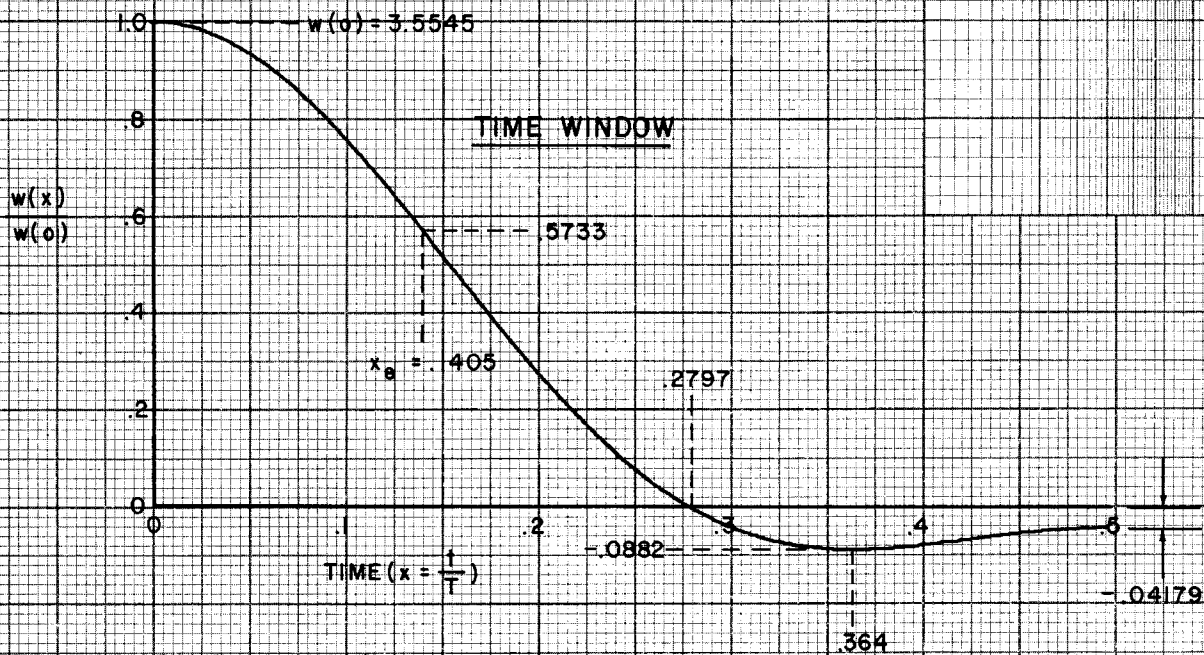
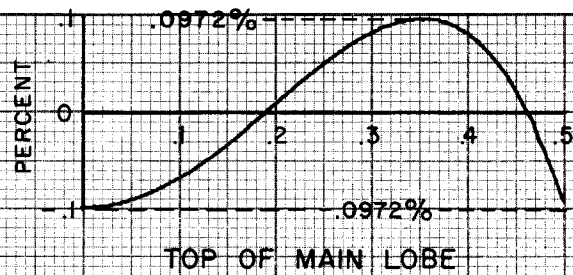
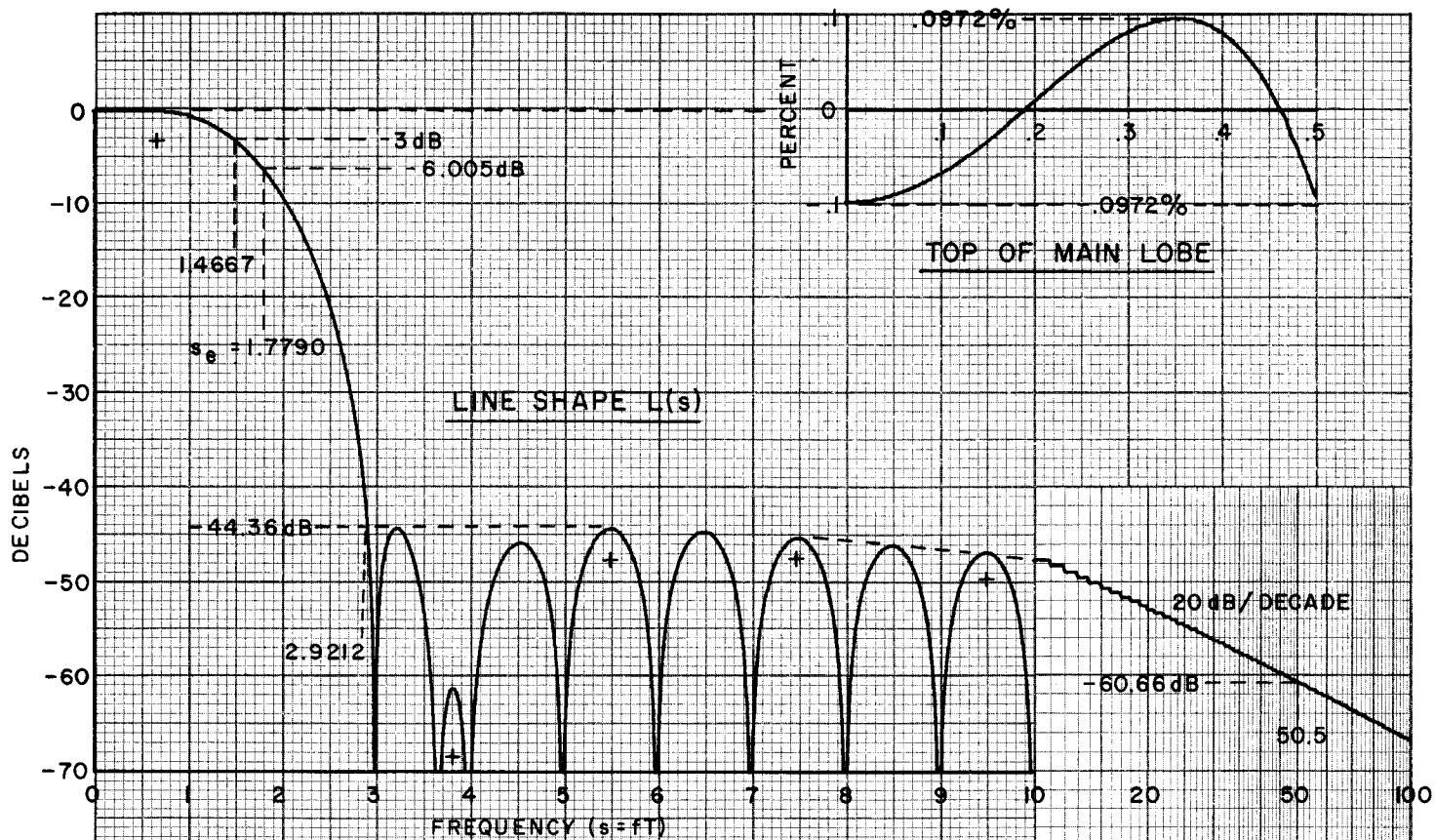
$$\text{Where: } a_0 = .999028 \quad a_2 = .351960$$

$$a_1 = .925752$$

$$\text{Polynomial Zeros: } s = \pm i1.412364, \pm 3.6722$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	3.5545
	(Line Shape) ²	2.9598
	Side Lobes	.3953
	(Side Lobes) ²	7.627×10^{-4}
Variance Reduction:	50% Time Record Overlap	.5063
	90% Time Record Overlap	.2862
	Quadratic Frequency Smoothing	.2849



Small side lobe amplitude and rapid roll off characteristics combine to produce a line shape with very low side lobe energy.

This window is most useful where large numbers of spectral lines are present, and leakage must be minimized. However, amplitude accuracy is rather poor (11%).

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-2}^2 a_k \delta(s-k), a_{-k} = a_k, N = 1024$$

$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^2 a_k \cos 2\pi kx \right]$$

$$\text{Where: } a_0 = 1 \qquad a_2 = .111290$$

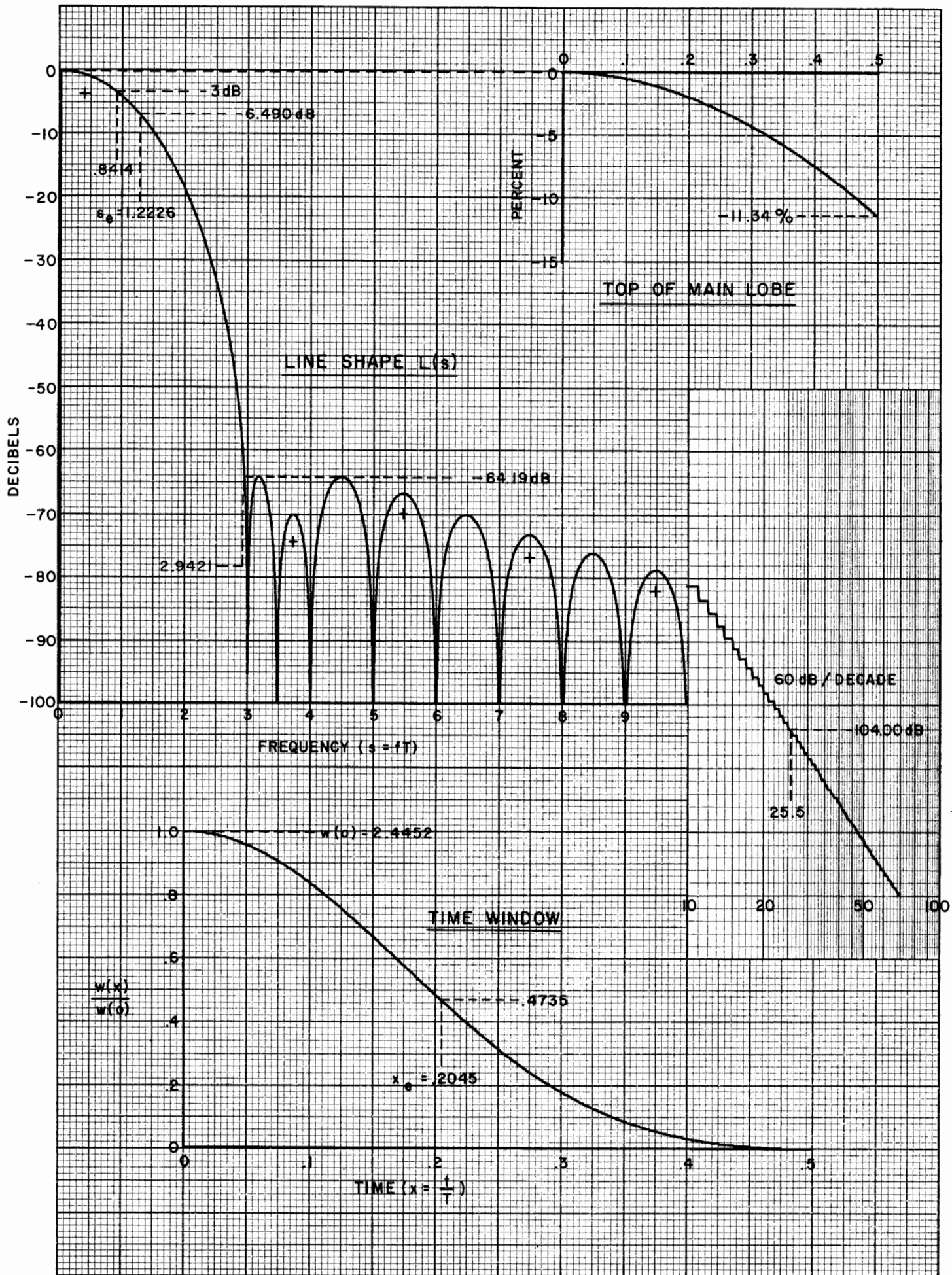
$$a_1 = .611290$$

$$\text{Polynomial Zeros: } s = \pm 3.4697$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	2.4452
	(Line Shape) ²	1.7721
	Side Lobes	4.890 x 10 ⁻³
	(Side Lobes) ²	1.587 x 10 ⁻⁶

Variance Reduction:	50% Time Record Overlap	.5061
	90% Time Record Overlap	.3771
	Quadratic Frequency Smoothing	.4075



This line shape is obtained by adjusting the locations of the first nine zeros to "level" the first nine side lobes. A rapid roll off rate is still preserved. The implementation of this shape is somewhat more involved because of the large number of coefficients, but the side lobe amplitude and energy are reduced considerably.

This line shape is used in place of P210 where the improved performance justifies the use of additional coefficients.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-10}^{10} a_k \delta(s-k), \quad a_{-k} = a_k, \quad N = 1024$$

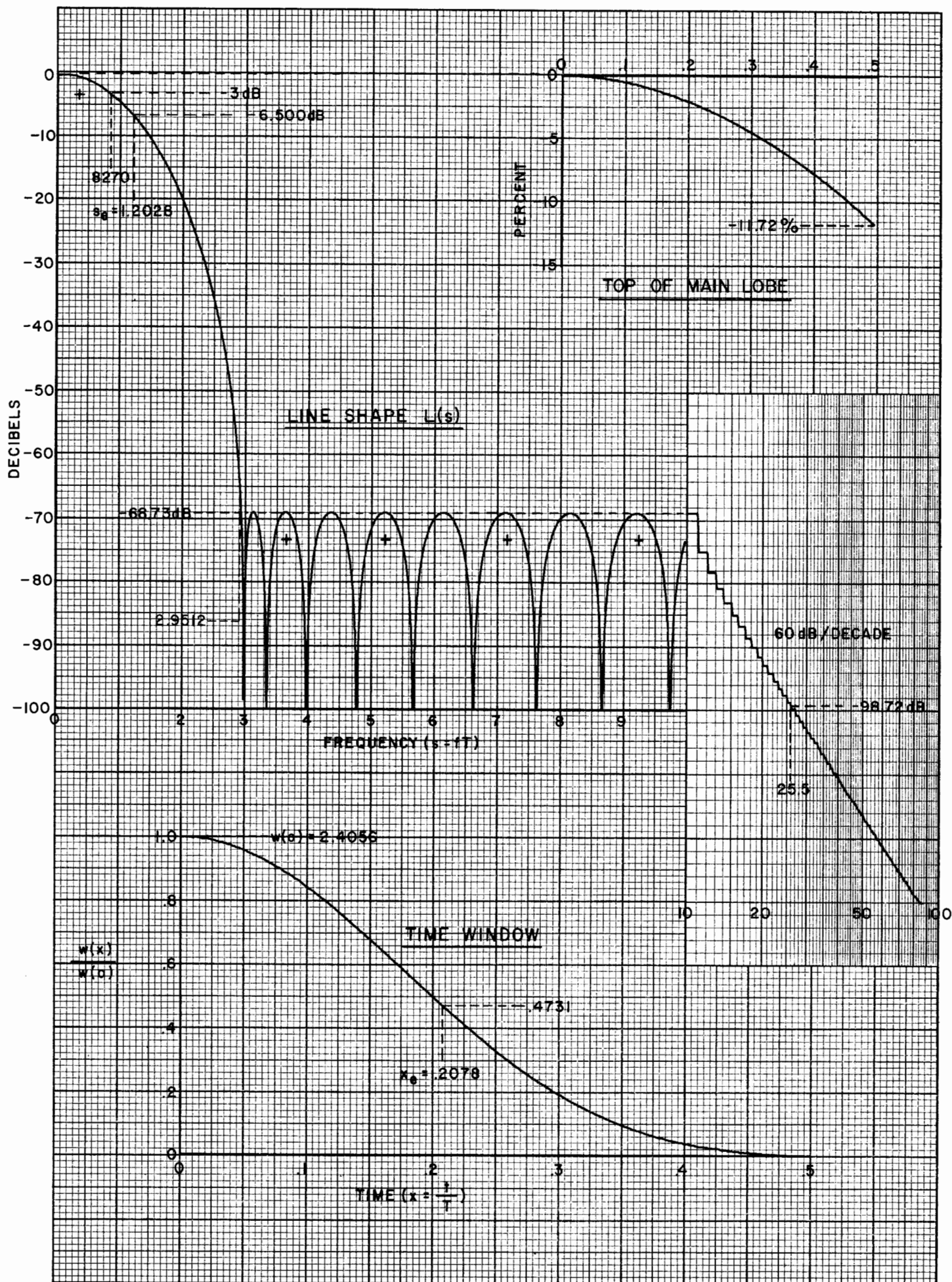
$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^{10} a_k \cos 2\pi kx \right]$$

$$\begin{aligned} \text{Where: } a_0 &= 1 & a_4 &= -2.56803 \times 10^{-6} & a_8 &= -3.22230 \times 10^{-4} \\ a_1 &= .600524 & a_5 &= 2.47449 \times 10^{-4} & a_9 &= 2.92653 \times 10^{-4} \\ a_2 &= .102252 & a_6 &= -3.14466 \times 10^{-4} & a_{10} &= -2.19609 \times 10^{-4} \\ a_3 &= 0 & a_7 &= 3.29447 \times 10^{-4} \end{aligned}$$

$$\begin{aligned} \text{Polynomial Zeros: } s &= \pm 3, \pm 3.36551, \pm 3.99837, \pm 4.79547, \pm 5.69056, \\ &\pm 6.64747, \pm 7.64819, \pm 8.68607, \pm 9.76908 \end{aligned}$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	2.4056
	(Line Shape) ²	1.7422
	Side Lobes	5.619 x 10 ⁻³
	(Side Lobes) ²	1.1170 x 10 ⁻⁶
Variance Reduction:	50% Time Record Overlap	.5074
	90% Time Record Overlap	.3819
	Quadratic Frequency Smoothing	.4152



This window is the result of Hanningtwice. Side lobes roll off extremely fast, but the first lobe is rather large. This shape is included because it is very easy to implement. However, other line shapes (such as P210) have considerably better characteristics.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-2}^2 a_k \delta(s-k), \quad a_{-k} = a_k, \quad N = 1024$$

$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^2 a_k \cos 2\pi kx \right]$$

$$\text{Where: } a_0 = 1 \quad a_2 = .166667$$

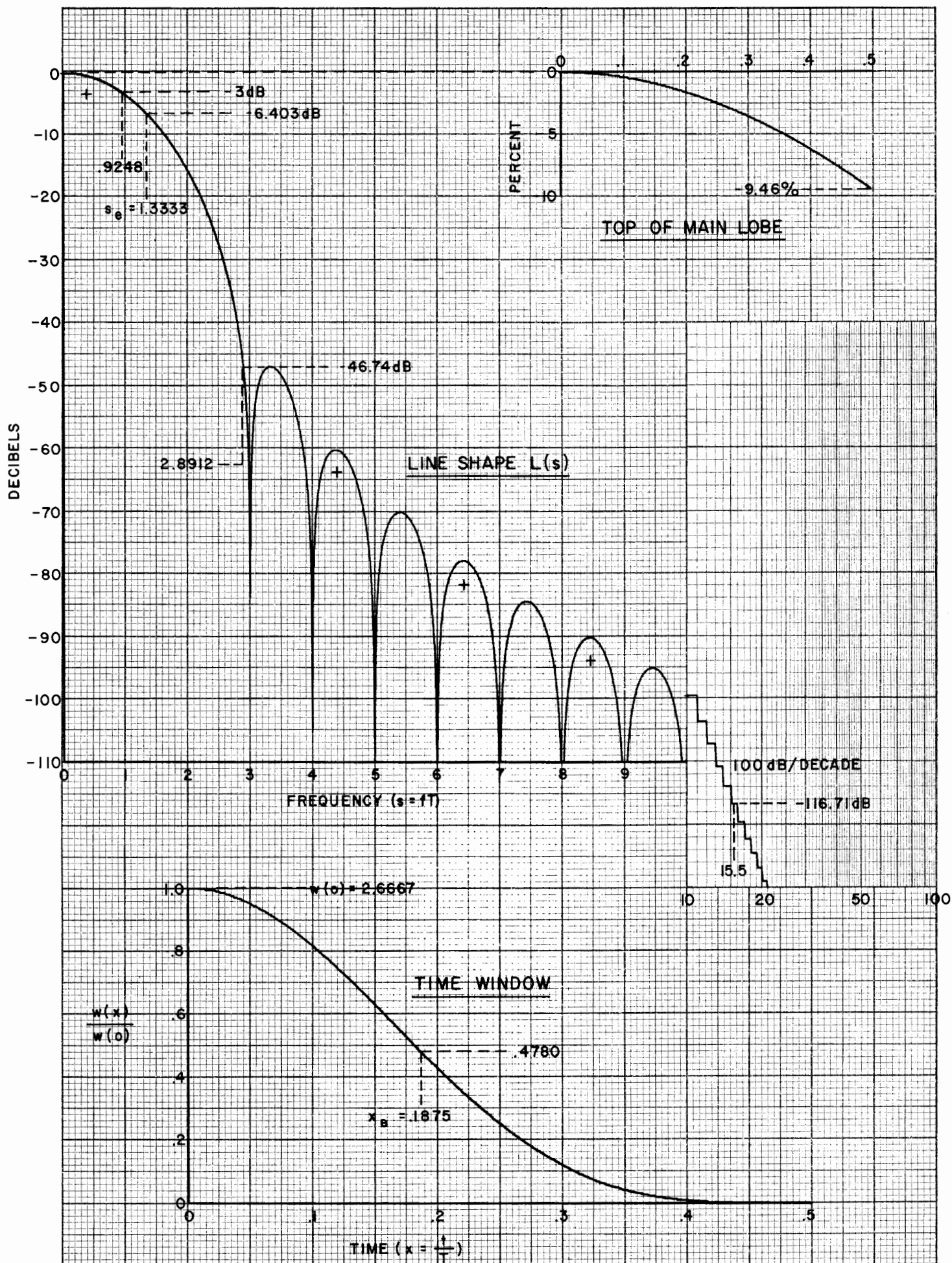
$$a_1 = .666667$$

There are no polynomial zeros.

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	2.66667
	(Line Shape) ²	1.94444
	Side Lobes	1.077 x 10 ⁻²
	(Side Lobes) ²	3.254 x 10 ⁻⁵

Variance Reduction:	50% Time Record Overlap	.5018
	90% Time Record Overlap	.3519
	Quadratic Frequency Smoothing	.3694



This shape is ideal for making accurate amplitude measurements ($\pm .1\%$) of spectral lines that are relatively close together, or where large dynamic range is needed. The main lobe top is very flat, and side lobes are very small, although the roll-off rate is slow.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-3}^3 a_k \delta(s-k), \quad a_{-k} = a_k, \quad N=1024$$

$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^3 a_k \cos 2\pi kx \right]$$

$$\text{Where } a_0 = .9994484 \quad a_2 = .539289$$

$$a_1 = .955728 \quad a_3 = .0915810$$

$$\text{Polynomial Zeros: } s = \pm i 1.62695, \pm 4.3593, \pm 6.5655$$

SUPPLEMENTAL CHARACTERISTICS

$$\text{Sample Areas: } \text{Line Shape} \quad 4.1726$$

$$(\text{Line Shape})^2 \quad 3.4242$$

$$|\text{Side Lobes}| \quad 3.675 \times 10^{-2}$$

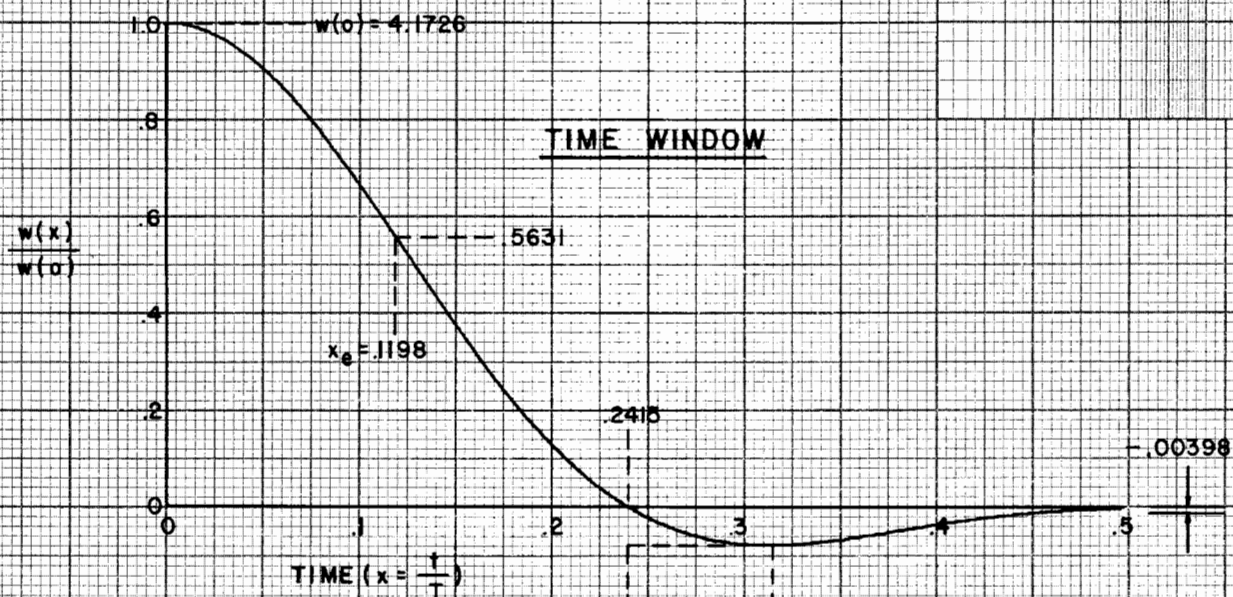
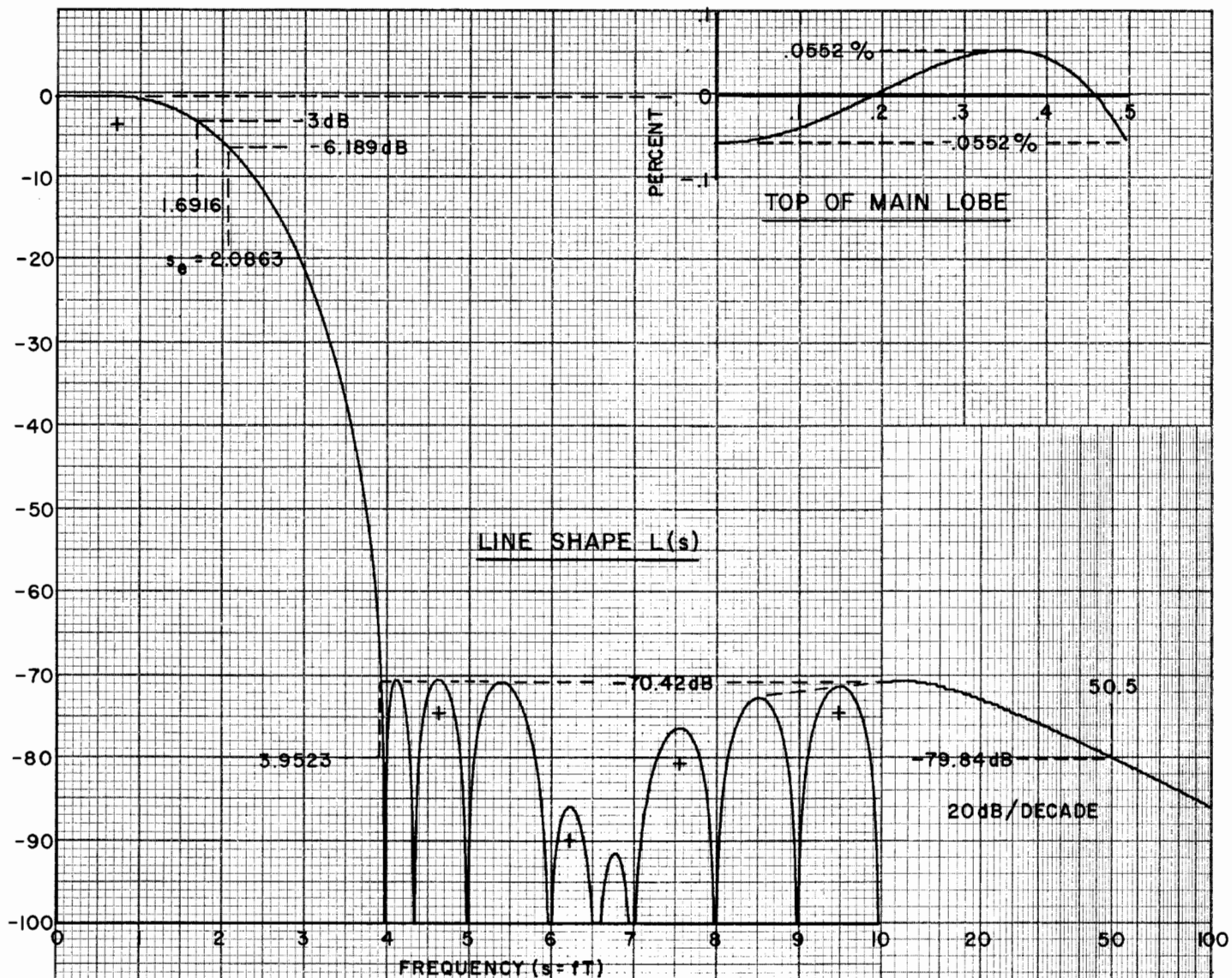
$$(\text{Side Lobes})^2 \quad 4.416 \times 10^{-6}$$

$$\text{Variance Reduction: } 50\% \text{ Time Record Overlap} \quad .5015$$

$$90\% \text{ Time Record Overlap} \quad .2432$$

$$\text{Quadratic Frequency Smoothing} \quad .2419$$

DECIBELS



This line shape is included to illustrate the dramatic reduction in sidelobe amplitude and energy that can be obtained with only a modest degree of main lobe broadening. As far as the HP Fourier Analyzer is concerned, the leakage from this line shape can be completely ignored.

This shape is useful where leakage would otherwise be a severe problem, but where the ultimate in frequency resolution is not needed, and where amplitude accuracy (9%) is of little concern.

DEFINITION

$$\text{Line Shape: } L(s) = \frac{\sin \pi s}{N \tan \frac{\pi s}{N}} * \sum_{k=-3}^3 a_k \delta(s-k), \quad a_{-k} = a_k, \quad N = 1024$$

$$\text{Time Window: } w(x) = \Pi(x) \cdot \left[a_0 + 2 \sum_{k=1}^3 a_k \cos 2\pi kx \right]$$

$$\begin{aligned} \text{Where: } a_0 &= 1 & a_2 &= .202701 \\ a_1 &= .684988 & a_3 &= .0177127 \end{aligned}$$

$$\text{Polynomial Zeros: } s = \pm 4.30535, \pm 5.3760$$

SUPPLEMENTAL CHARACTERISTICS

Sample Areas:	Line Shape	2.8108
	(Line Shape) ²	2.0212
	Side Lobes	3.787 x 10 ⁻⁴
	(Side Lobes) ²	4.846 x 10 ⁻⁹

Variance Reduction:	50% Time Record Overlap	.50125
	90% Time Record Overlap	.3391
	Quadratic Frequency Smoothing	.3534

DECIBELS

