

Effective Measurements for Structural Dynamics Testing

PART I

Kenneth A. Ramsey, Hewlett-Packard Company, Santa Clara, California

Digital Fourier analyzers have opened a new era in structural dynamics testing. The ability of these systems to quickly and accurately measure a set of structural frequency response functions and then operate on them to extract modal parameters is having a significant impact on the product design and development cycle. Part I of this article is intended to introduce the structural dynamic model and the representation of modal parameters in the Laplace domain. The concluding section explains the theory for measuring structural transfer functions with a digital analyzer. Part II will be directed at presenting various practical techniques for measuring these functions with sinusoidal, transient and random excitation. New advances in random excitation will be presented and digital techniques for separating closely coupled modes via increased frequency resolution will be introduced.

Structural Dynamics and Modal Analysis

Understanding the dynamic behavior of structures and structural components is becoming an increasingly important part of the design process for any mechanical system. Economic and environmental considerations have advanced to the state where over-design and less than optimum performance and reliability are not readily tolerated. Customers are demanding products that cost less, last longer, are less expensive to operate, while at the same time they must carry more pay-load, run quieter, vibrate less, and fail less frequently. These demands for improved product performance have caused many industries to turn to advanced structural dynamics testing technology.

The use of experimental structural dynamics as an integral part of the product development cycle has varied widely in different industries. Aerospace programs were among the first to apply these techniques for predicting the dynamic performance of flight vehicles. This type of effort was essential because of the weight, safety, and performance constraints inherent in aerospace vehicles. Recently, increased consumer demand for fuel economy, reliability, and superior vehicle ride and handling qualities have been instrumental in making structural dynamics testing an integral part of the automotive design cycle. An excellent example was reported in the cover story article on the new Cadillac Seville from Automotive Industries, April 15, 1975.

"The most radical use of computer technology which 'will revolutionize the industry' is dynamic structural analysis, or Fourier analysis as it is commonly known. It was this technique, in con-

junction with others, that enabled Cadillac to 'save a mountain of time and money,' and pare down the number of prototypes necessary. It also did away with much trial and error on the solution of noise and vibration problems."

In order to understand the dynamic behavior of a vibrating structure, measurements of the dynamic properties of the structure and its components are essential. Even though the dynamic properties of certain components can be determined with finite computer techniques, experimental verification of these results are still necessary in most cases.

One area of structural dynamics testing is referred to as modal analysis. Simply stated, *modal analysis* is the process of characterizing the dynamic properties of an elastic structure by identifying its modes of vibration. That is, each mode has a specific natural frequency and damping factor which can be identified from practically any point on the structure. In addition, it has a characteristic "mode shape" which defines the mode spatially over the entire structure.

Once the dynamic properties of an elastic structure have been characterized, the behavior of the structure in its operating environment can be predicted and, therefore, controlled and optimized.

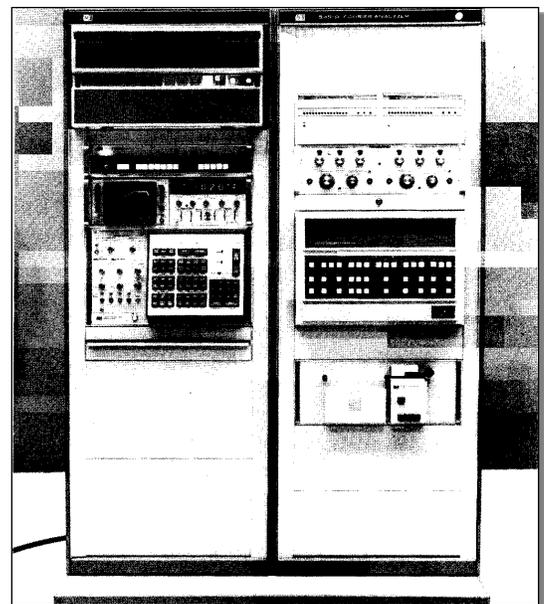


Figure 1—The HP 5451 B Fourier Analyzer is typical of modern digital analyzers that can be used for acquisition and processing of structural dynamics data.

In general, modal analysis is valuable for three reasons:

- 1) Modal analysis allows the verification and adjusting of the mathematical models of the structure. The equations of motion are based on an idealized model and are used to predict and simulate dynamic performance of the structure. They also allow the designer to examine the effects of changes in the mass, stiffness, and damping properties of the structure in greater detail. For anything except the simplest structures, modeling is a formidable task. Experimental measurements on the actual hardware result in a physical check of the accuracy of the mathematical model. If the model predicts the same behavior that is actually measured, it is reasonable to extend the use of the model for simulation, thus reducing the expense of building hardware and testing each different configuration. This type of modeling plays a key role in the design and testing of aerospace vehicles and automobiles, to name only two.
- 2) Modal analysis is also used to locate structural weak points. It provides added insight into the most effective product design for avoiding failure. This often eliminates the tedious trial and error procedures that arise from trying to apply inappropriate static analysis techniques to dynamic problems.
- 3) Modal analysis provides information that is essential in eliminating unwanted noise or vibration. By understanding how a structure deforms at each of its resonant frequencies, judgments can be made as to what the source of the disturbance is, what its propagation path is, and how it is radiated into the environment.

In recent years, the advent of high performance, low cost minicomputers, and computing techniques such as the fast Fourier transform have given birth to powerful new "instruments" known as digital Fourier analyzers (see Figure 1). The ability of these machines to quickly and accurately provide the frequency spectrum of a time-domain signal has opened a new era in structural dynamics testing. It is now relatively simple to obtain fast, accurate, and complete measurements of the dynamic behavior of mechanical structures, via transfer function measurements and modal analysis.

Techniques have been developed which now allow the modes of vibration of an elastic structure to be identified from measured transfer function data,^{1,2}. Once a set of transfer (frequency response) functions relating points of interest on the structure have been measured and stored, they may be operated on to obtain the modal parameters; i.e., the natural frequency, damping factor, and characteristic mode shape for the predominant modes of vibration of the structure. Most importantly, the modal responses of many modes can be measured simultaneously and complex mode shapes can be directly identified, permitting one to avoid attempting to isolate the response of one mode at a time, i.e., the so called "normal mode" testing concept.

The purpose of this article is to address the problem of making effective structural transfer function measurements for modal analysis. First, the concept of a transfer function will be explored. Simple examples of one and two degree of freedom models will be used to explain the representation of a mode in the Laplace domain. This representation is the key to understanding the basis for extracting modal parameters from measured data. Next, the digital computation of the transfer function will be shown. In Part II, the advantages and disadvantages of various excitation types and a comparison of results will illustrate the importance of choosing the proper type of excitation. In addition, the solution for the problem of inadequate frequency resolution, non-linearities and distortion will be presented.

The Structural Dynamics Model

The use of digital Fourier analyzers for identifying the modal properties of elastic structures is based on accurately measuring structural transfer (frequency response) functions. This measured data contains all of the information necessary for obtaining the modal (Laplace) parameters which completely define the structures' modes of vibration. Simple one and two degree of freedom lumped models are effective tools for introducing the concepts of a transfer function, the s-plane representation of a mode, and the corresponding modal parameters.

The idealized single degree of freedom model of a simple vibrating system is shown in Figure 2. It consists of a spring, a damper, and a single mass which is constrained to move along one axis only. If the system behaves linearly and the mass is subjected to any arbitrary time varying force, a corresponding time varying motion, which can be described by a linear second order ordinary differential equation, will result. As this motion takes place, forces are generated by the spring and damper as shown in Figure 2.

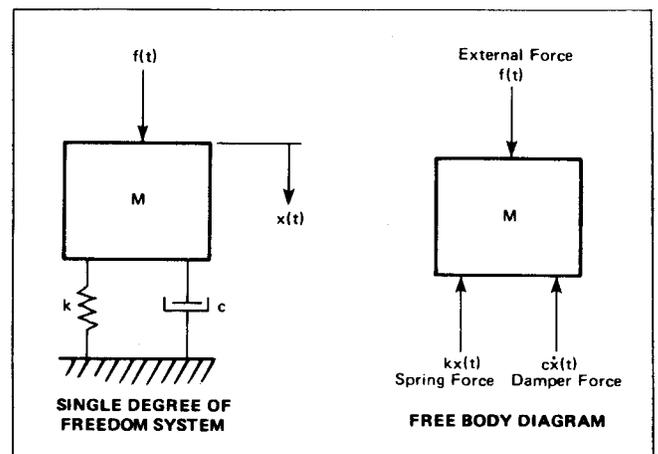


Figure 2—Idealized single degree of freedom model.

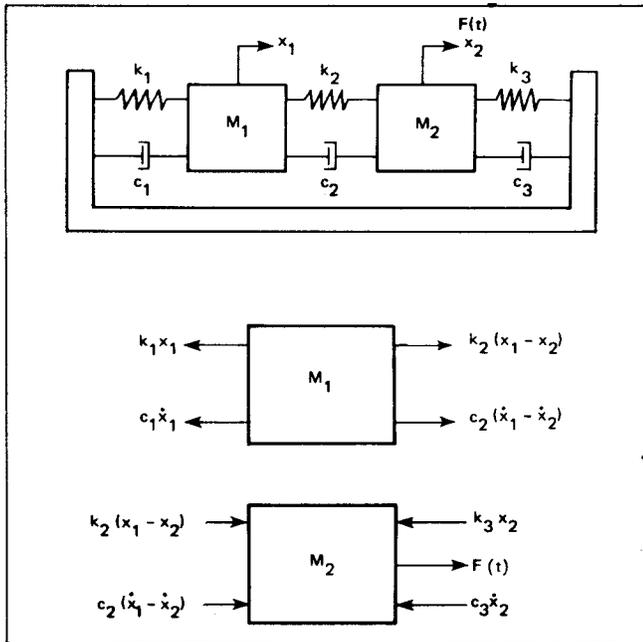


Figure 3—A two degree of freedom model.

The equation of motion of the mass m is found by writing Newton's second law for the mass ($\sum F_{ext} = ma$), where ma is a real inertial force,

$$f(t) - kx(t) - c\dot{x}(t) = m\ddot{x}(t) \quad (1)$$

where $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second time derivatives of the displacement $x(t)$. Rewriting equation (1) results in the more familiar form:

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad (2)$$

where:
 $f(t)$ = applied force
 x = resultant displacement
 \dot{x} = resultant velocity
 \ddot{x} = resultant acceleration

and m , c , and k are the mass, damping constant, and spring constant, respectively. Equation (2) merely balances the inertia force ($m\ddot{x}$), the damping force ($c\dot{x}$), and the spring force (kx), against the externally applied force, $f(t)$.

The multiple degree of freedom case follows the same general procedure. Again, applying Newton's second law, one may write the equations of motion as:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0 \quad (3)$$

and

$$m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_2 + (k_2 + k_3)x_2 - c_2\dot{x}_1 - k_2x_1 = F(t) \quad (4)$$

It is often more convenient to write equations (3) and (4) in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & (-c_2) \\ (-c_2) & (c_2 + c_3) \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (-k_2) \\ (-k_2) & (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} \quad (5)$$

or equivalently, for the general n -degree of freedom system,

$$[M] \{\ddot{x}\} + [C] \{\dot{x}\} + [K] \{x\} = \{F\} \quad (6)$$

Where, $[M]$ = mass matrix, ($n \times n$),
 $[C]$ = damping matrix, ($n \times n$),
 $[K]$ = stiffness matrix, ($n \times n$),

and the previously defined force, displacement, velocity, and acceleration terms are now n -dimensional vectors.

The mass, stiffness, and damping matrices contain all of the necessary mass, stiffness, and damping coefficients such that the equations of motion yield the correct time response when arbitrary input forces are applied.

The time-domain behavior of a complex dynamic system represented by equation (6) is very useful information. However, in a great many cases, frequency domain information turns out to be even more valuable. For example, natural frequency is an important characteristic of a mechanical system, and this can be more clearly identified by a frequency domain representation of the data. The choice of domain is clearly a function of what information is desired.

One of the most important concepts used in digital signal processing is the ability to transform data between the time and frequency domains via the Fast Fourier Transform (FFT) and the Inverse FFT. The relationships between the time, frequency, and Laplace domains are well defined and greatly facilitate the process of implementing modal analysis on a digital Fourier analyzer. Remember that the Fourier and Laplace transforms are the mathematical tools that allow data to be transformed from one independent variable to another (time, frequency or the Laplace s -variable). The discrete Fourier transform is a mathematical tool which is easily implemented in a digital processor for transforming time-domain data to its equivalent frequency domain form, and vice versa. It is important to note that no information about a signal is either gained or lost as it is transformed from one domain to another.

The transfer (or characteristic) function is a good example of the versatility of presenting the same information in three different domains. In the time domain, it is the unit impulse response, in the frequency domain the frequency response function and in the Laplace or s-domain, it is the transfer function. Most importantly, all are transforms of each other.

Because we are concerned with the identification of modal parameters from transfer function data, it is convenient to return to the single degree of freedom system and write equation (2) in its equivalent transfer function form.

The Laplace Transform. Recall that a function of time may be transformed into a function of the complex variable s by:

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \tag{7}$$

The Laplace transform of the equation of motion of a single degree of freedom system, as given in equation (2), is

$$m[s^2 X(s) - sx(0) - \dot{x}(0)] + c[sX(s) - x(0)] + kX(s) = F(s) \tag{8}$$

where,

$x(0)$ is the initial displacement of the mass m and

$\dot{x}(0)$ is the initial velocity.

This transformed equation can be rewritten by combining the initial conditions with the forcing function, to form a new $F(s)$:

$$[ms^2 + cs + k] X(s) = F(s) \tag{9}$$

It should now be clear that we have transformed the original ordinary differential equation into an algebraic equation where s is a complex variable known as the Laplace operator. It is also said that the problem is transformed from the time (real) domain into the s (complex) domain, referring to the fact that time is always a real variable, whereas the equivalent information in the s-domain is described by complex functions. One reason for the transformation is that the mathematics are much easier in the s-domain. In addition, it is generally easier to visualize the parameters and behavior of damped linear systems in the s-domain.

Solving for X(s) from equation (9), we find

$$X(s) = \frac{F(s)}{ms^2 + cs + k} \tag{10}$$

The denominator polynomial is called the characteristic equation, since the roots of this equation determine the character of the time response. The roots of this charac-

teristic equation are also called the *poles* or *singularities* of the system. The roots of the numerator polynomial are called the *zeros* of the system. Poles and zeros are critical frequencies. At the poles the function $x(s)$ becomes infinite; while at the zeros, the function becomes zero. A transfer function of a dynamic system is defined as the ratio of the output of the system to the input in the s-domain. It is, by definition, a function of the complex variable s. If a system has m inputs and n resultant outputs, then the system has $m \times n$ transfer functions. The transfer function which relates the displacement to the force is referred to as the compliance transfer function and is expressed mathematically as,

$$H(s) = \frac{X(s)}{F(s)} \tag{11}$$

From equations (10) and (11), the compliance transfer function is,

$$H(s) = \frac{1}{ms^2 + cs + k} \tag{12}$$

Note that since s is complex, the transfer function has a real and an imaginary part. The Fourier transform is obtained by merely substituting $j\omega$ for s. This special case of the transfer function is called the *frequency response function*. In other words, the Fourier transform is merely the Laplace transform evaluated along the $j\omega$, or frequency axis, of the complex Laplace plane.

The analytical form of the frequency response function is therefore found by letting $s = j\omega$:

$$H(j\omega) = \frac{1}{-m\omega^2 + jc\omega + k} \tag{13}$$

By making the following substitutions in equation (13),

$$\omega_n^2 = \frac{k}{m}, \quad \zeta = \frac{c}{C_c} = \frac{c}{2\sqrt{km}}$$

$C_c =$ critical damping coefficient

we can write the classical form of the frequency response function so,

$$\frac{X(j\omega)}{F(j\omega)} = H(j\omega) = \frac{1}{k \left[1 + 2\zeta j \left(\frac{\omega}{\omega_n} \right) - \frac{\omega^2}{\omega_n^2} \right]} \tag{14}$$

However, for our purposes, we will continue to work in the s-domain. The above generalized transfer function, equation (12), was developed in terms of compliance.

From an experimental viewpoint, other very useful forms of the transfer function are often used and, in general,

Table I – Different forms of the transfer function for mechanical systems.

$\frac{\text{Displacement}}{\text{Force}}$	Dynamic Compliance	$\frac{\text{Force}}{\text{Displacement}}$	Dynamic Stiffness
=		=	
$\frac{\text{Velocity}}{\text{Force}}$	Mobility	$\frac{\text{Force}}{\text{Velocity}}$	Mechanical Impedance
=		=	
$\frac{\text{Acceleration}}{\text{Force}}$	Inertance	$\frac{\text{Force}}{\text{Acceleration}}$	Dynamic Mass
=		=	

contain the same information. Table I summarizes these different forms.

The s-Plane. Since s is a complex variable, we can represent all complex values of s by points in a plane. Such a plane is referred to as the s -plane. Any complex value of s may be located by plotting its real component on one axis and its imaginary component on the other. Now, the magnitude of any function, such as the compliance transfer function, $H(s)$, can be plotted as a surface above the plane of Figure 4. This requires a three-dimensional figure which can be difficult to sketch, but greatly facilitates the understanding of the transfer function. By definition, $s = \sigma + j\omega$ where σ is the *damping coefficient* and ω is the *angular frequency*.

The inertance transfer function of a simple two degree of freedom system is plotted as a function of the s variable in Figure 5. The transfer function evaluated along the frequency axis ($s = j\omega$) is the Fourier transform or the system frequency response function. It is shown by the heavy line. If we were to measure the frequency response function for this system via experimental measurements using the Fourier transform, we would obtain a complex-valued function of frequency. It must be represented by its real (coincident) part and its imaginary (quadrature) part; or equivalently, by its magnitude and phase. These forms are shown in Figure 6.

In general, complex mechanical systems contain many modes of vibration or "degrees of freedom." Modern modal analysis techniques can be used to extract the modal parameters of each mode without requiring each mode to be isolated or excited by itself.

Modes of Vibration The equations of motion of an n degree of freedom system can be written as

$$B(s) X(s) = F(s) \tag{15}$$

Where, $F(s)$ = Laplace transform of the applied force vector

$X(s)$ = Laplace transform of the

resulting output vector

$$B(s) = Ms^2 + Cs + K$$

s = Laplace operator

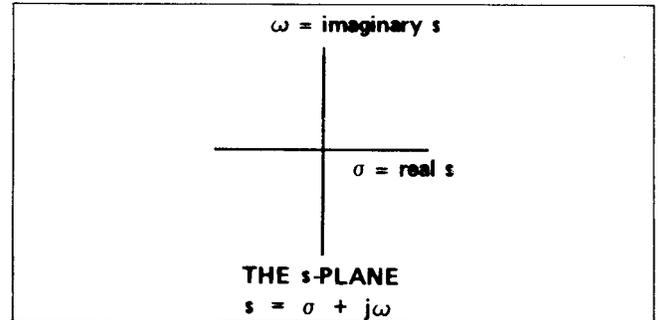


Figure 4—The s plane, $s = \sigma + j\omega$

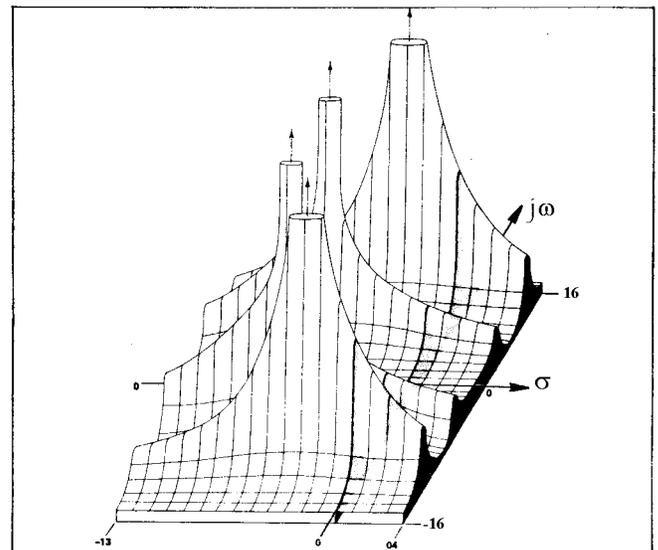


Figure 5—The magnitude of the Laplace transform of a two degree of freedom system with poles at $s = -3 \pm j11$ and $s = -6 \pm j6$.

$B(s)$ is referred to as the system matrix. The transfer matrix, $H(s)$ is defined as the inverse of the system matrix, hence it satisfies the equation.

$$X(s) = H(s) F(s) \tag{16}$$

Each element of the transfer matrix is a transfer function.

From the general form of the transfer function described in equation (16), $H(s)$ can always be written in partial fraction form as:

$$H(s) = \sum_{k=1}^{2n} \frac{a_k}{s - p_k} \tag{17}$$

Figure 6—Frequency response functions of a two degree of freedom system.

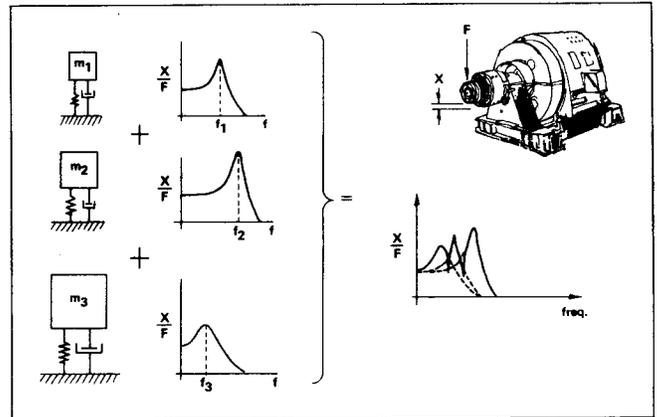
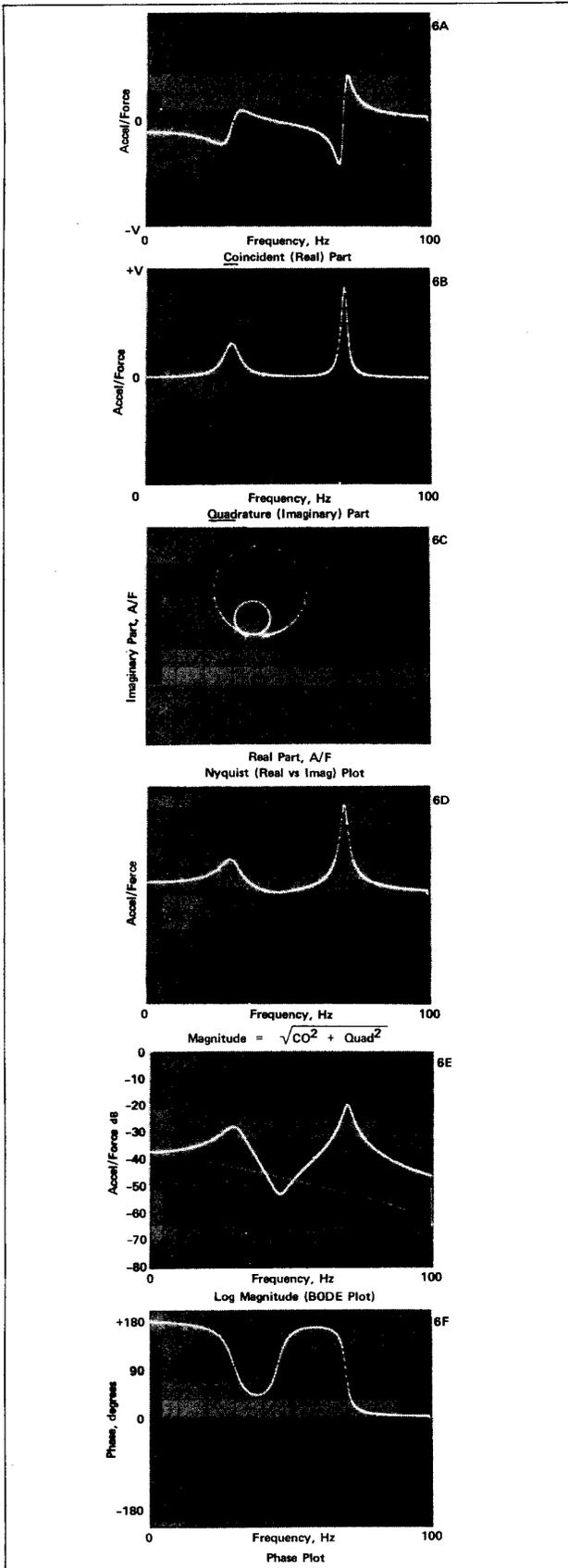


Figure 7—A multi-degree of freedom system can be ideally represented as a series of coupled single degree of freedom systems.



where, n = number of degrees of freedom

$p_k = k_{th}$ root of the equation obtained by setting the determinant of the matrix $B(s)$ equal to zero

a_k = residue matrix for the k th root.

As mentioned earlier, the roots p_k are referred to as poles of the transfer function. These poles are complex numbers and always occur in complex conjugate pairs, except when the system is critically or supercritically damped. In the latter cases, the poles are real-valued and lie along the real (or damping) axis in the s -plane.

Each complex conjugate pair of poles corresponds to a *mode of vibration* of the structure. They are complex numbers written as

$$p_k = -\sigma_k + j\omega_k \quad p_k^* = -\sigma_k - j\omega_k \quad (18)$$

Where $*$ denotes the conjugate, σ_k is the *modal damping coefficient*, and ω_k is the *natural frequency*. These parameters are shown on the s -plane in Figure 8. An alternate set of coordinates for defining the pole locations are the *resonant frequency*, given by

$$\Omega_k = \sqrt{\sigma_k^2 + \omega_k^2} \quad (19)$$

and the *damping factor*, or *percent of critical damping*, given by:

$$\zeta_k = \frac{\sigma_k}{\Omega_k} \quad (20)$$

The transfer matrix completely defines the dynamics of the system. In addition to the poles of the system (which define the natural frequency and damping), the residues from any row or column of $H(s)$ define the system

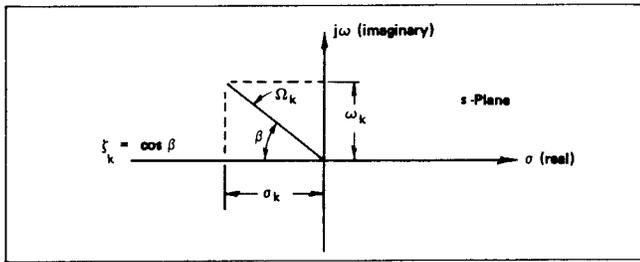


Figure 8—An s-plane representation for a single degree of freedom system.

mode shapes for the various natural frequencies. In general, a pole location, P_k , will be the same for all transfer functions in the system because a mode of vibration is a global property of an elastic structure. The values of the residues, however, depend on the particular transfer function being measured. The values of the residues determine the amplitude of the resonance in each transfer function and, hence, the mode shape for the particular resonance. From complex variable theory, we know that if we can measure the frequency response function (via the Fourier transform) then we know the exact form of the system (its transfer function) in the s-plane, and hence we can find the four important properties of any mode. Namely, its natural frequency, damping, and magnitude and phase of its residue or amplitude.

While this is a somewhat trivial task for a single degree of freedom system, it becomes increasingly difficult for complex systems with many closely coupled modes. However, considerable effort has been spent in recent years to develop sophisticated algorithms for curve-fitting to experimentally measured frequency response functions.^{1,2} This allows the modal properties of each measured mode to be extracted in the presence of other modes.

From a testing standpoint, these new techniques offer important advantages. Writing equation (16) in matrix form gives:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix} \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix} \quad (21)$$

If only one mode is associated with each pole, then it can be shown that the modal parameters can be identified from any row or column of the transfer function matrix $[H]$, except those corresponding to components known as node points. In other words, it is impossible to excite a mode by forcing it at one of its node points (a point where no response is present). Therefore, only one row or column need be measured.

To measure one column on the transfer matrix, an exciter would be attached to the structure (point #1 to measure column #1; point #2 to measure column #2) and responses would be measured at points #1 and #2. Then the transfer function would be formed by computing,

$$H(j\omega) = \frac{X(j\omega)}{F(j\omega)} \quad (22)$$

To measure a row of the transfer matrix, the structure would be excited at point #1 and the response measured at point #1. Next, the structure would be excited at point #2 and the response again measured at point #1. This latter case corresponds to having a stationary response transducer at point #1, and using an instrumented hammer for applying impulsive forcing functions. Both of these methods are referred to as single point excitation techniques.

Complex Mode Shapes. Before leaving the structural dynamic model, it is important to introduce the idea of a complex mode shape. Without placing restrictions on damping beyond the fact that the damping matrix be symmetric and real valued, modal vectors can in general be complex valued. When the mode vectors are real valued, they are the equivalent of the *mode shape*. In the case of complex modal vectors, the interpretation is slightly different.

Recall that the transfer matrix for a single mode can be written as;

$$H_k(s) = \frac{a_k}{s - p_k} + \frac{a_k^*}{s - p_k^*} \quad (23)$$

where

$a_k = (n \times n)$ complex residue matrix.

$p_k =$ pole location of mode k .

A single component of $H(s)$ is thus written as

$$H(s) = \frac{r_k}{2j(s - p_k)} - \frac{r_k^*}{2j(s - p_k^*)} \quad (24)$$

where

$$\frac{r_k}{2j} = \text{complex residue of mode } k.$$

Now, the inverse Laplace transform of the transfer function of equation (24) is the *impulse response* of mode k ; that is, if only mode k was excited by a unit impulse, its time domain response would be

$$x_k(t) = |r_k| e^{-\sigma_k t} \sin(\omega_k t - \alpha_k) \quad (25)$$

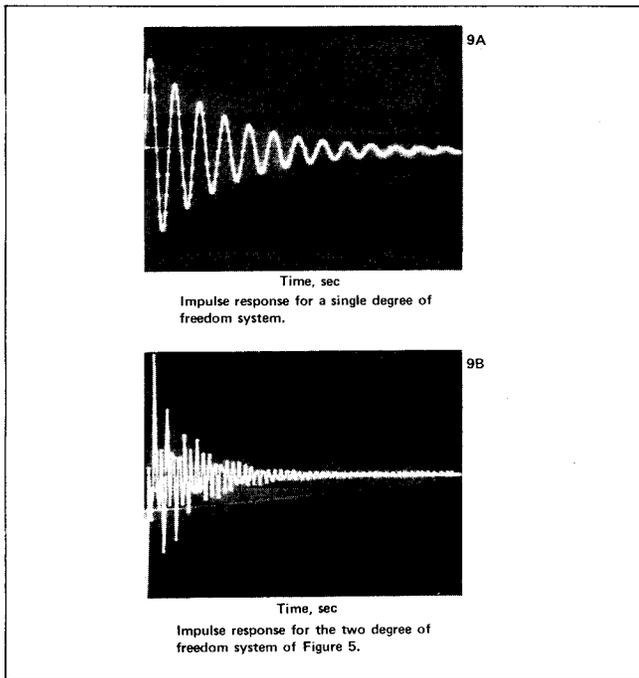


Figure 9—Impulse response for a single degree of freedom system and for the two degree of freedom system shown in Figure 5.

where

$$|r_k| = \text{magnitude of the residue}$$

$$\alpha_k = \text{phase angle of the residue}$$

A phase shift in the impulse response is introduced by the phase angle α_k of the complex residue. For $\alpha_k = 0$, the mode is said to be "normal" or real valued. It is this phase delay in the impulse response that is represented by the complex mode shape. Experimentally, a real or normal mode is characterized by the fact that all points on the structure reach their maximum or minimum deflection at the same time. In other words, all points are either in phase or 180° out of phase. With a complex mode, phases other than 0° and 180° are possible. Thus, nodal lines will be stationary for normal modes and nonstationary or "traveling" for complex modes. The impulse response for a single degree of freedom system and for the two degree of freedom system represented in Figure 5 are shown in Figure 9.

The digital Fourier Analyzer has proven to be an ideal tool for measuring structural frequency response functions quickly and accurately. Since it provides a broadband frequency spectrum very quickly (e.g., = 100 ms for 512 spectral lines when implemented in microcode), it can be used for obtaining broadband response spectrums from a structure which is excited by a broadband input signal. Furthermore, if the input and response time signals are measured simultaneously, Fourier transformed, and the transform of the response is divided by the transform of the input, a transfer function between the input

and response points on the structure is measured. Because the Fourier Analyzer contains a digital processor, it possesses a high degree of flexibility in being able to post-process measured data in many ways.

It has been shown^{1,2} that the modes of vibration of an elastic structure can be identified from transfer function measurements by the application of digital parameter identification techniques. Hewlett-Packard has implemented these techniques on the HP 5451B Fourier Analyzer. The system uses a single point excitation technique. This approach, when coupled with a broadband excitation allows all modes in the bandwidth of the input energy to be excited simultaneously. The modal frequencies, damping coefficients, and residues (eigenvectors) are then extracted from the measured broadband transfer functions via an analytical curve-fitting algorithm. This method thus permits an accurate definition of modal parameters without exciting each mode individually. Part II of this article will address the problem of making transfer function measurements.

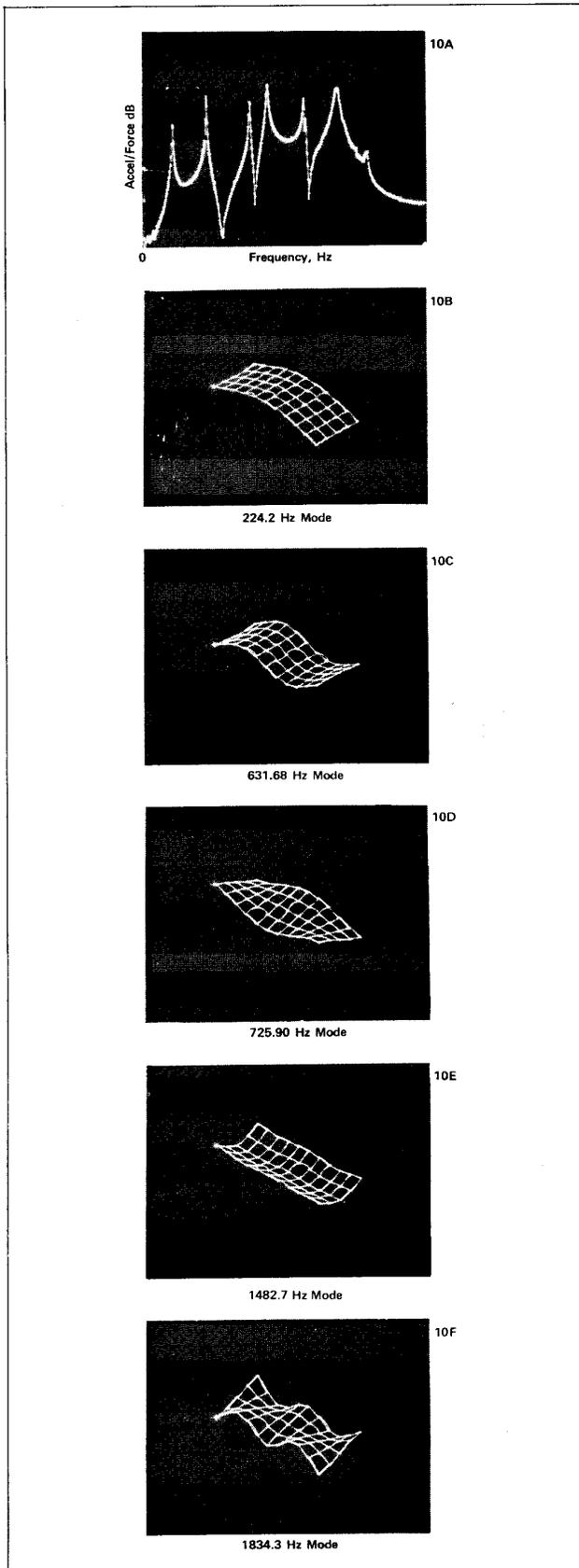
The data shown in Figure 10 was obtained by using the Hewlett-Packard HP 5451B Fourier Analyzer to measure the required set of frequency response functions from a simple rectangular plate and identify the predominant modes of vibration. Figure 10A shows a typical frequency response function obtained from using an impulse testing technique on a flat aluminum plate. Input force was measured with a load cell and the output response was measured with an accelerometer. After 55 such functions were measured and stored, the modal parameters were identified via a curve-fitting algorithm. In addition, the Fourier Analyzer provided an animated isometric display of each mode, the results of which are shown in Figures 10B – 10F.

The Transfer and Coherence Functions

The measurement of structural transfer functions using digital Fourier analyzers has many important advantages for the testing laboratory. However, it is imperative that one have a firm understanding of the measurement process in order to make effective measurements. For instance, digital techniques require that all measurements be discrete and of finite duration. Thus, in order to implement the Fourier transform digitally, it must be changed to a finite form known as the Discrete Fourier Transform (DFT). This means that all continuous time waveforms which must be transformed must be sampled (measured) at discrete intervals of time, uniformly separated by an interval Δt . It also means that only a finite number of samples N can be taken and stored. The record length T is then

$$T = N\Delta t \tag{26}$$

The effect of implementing the DFT in a digital memory is that it no longer contains magnitude and phase information at all frequencies as would be the ease for the continuous Fourier transform.



Rather, it describes the spectrum of the waveform at discrete frequencies and with finite resolution up to some maximum frequency, F_{max} , which according to Shannon's sampling theorem, obeys

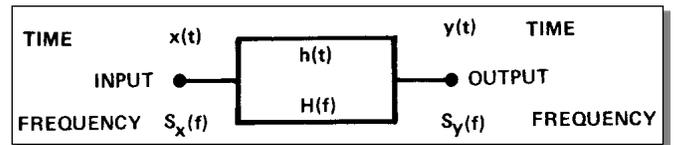
$$F_{max} = \frac{1}{2\Delta t} \tag{27}$$

As a direct consequence of equation (27), we can write the physical law which defines the maximum frequency resolution obtainable for a sampled record of length, T.

$$\Delta f = \frac{1}{T} \tag{28}$$

When dealing with real valued-time functions, there will be N points in the record. However, to completely describe a given frequency, two values are required; the magnitude and phase or, equivalently, the real part and the imaginary part. Consequently, N points in the time domain can yield $N/2$ complex quantities in the frequency domain. With these important relationships in mind, we can return to the problem of measuring transfer functions.

The general case for a system transfer function measurement is shown below



where:

$x(t)$ = Time-domain input to the system

$y(t)$ = Time-domain response of the system

$S_x(f)$ = Linear Fourier spectrum of $x(t)$

$S_y(f)$ = Linear Fourier spectrum of $y(t)$

$H(f)$ = System transfer function (frequency domain)

$h(t)$ = System impulse response

The linear Fourier spectrum is a complex valued function that results from the Fourier transform of a time waveform. Thus, S_x and S_y have a real (in phase or coincident) and imaginary (quadrature) parts.

In general, the result of a linear system on any time domain input signal, $x(t)$, may be determined from the convolution of the system impulse response, $h(t)$, with the input signal, $x(t)$, to give the output, $y(t)$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \tag{29}$$

Figure 10—Typical frequency response function of the vibration of a simple rectangular plate (A) and animated isometric displays of the predominant vibration modes (B-F).

This operation may be difficult to visualize. However, a very simple relationship can be obtained by applying the Fourier transform to the convolution integral. The output spectrum, S_y , is the product of the input spectrum, S_x , and the system transfer function, $H(f)$.

$$S_y(f) = S_x(f) \cdot H(f) \tag{30}$$

In other words, the transfer function of the system is defined as:

$$H = \frac{\text{OUTPUT}}{\text{INPUT}} = \frac{S_y}{S_x} \tag{31}$$

The simplest implementation of a measurement scheme based on this technique is the use of a sine wave for $x(f)$. However, in many cases, this signal has disadvantages compared to other more general types of signals. The most general method is to measure the input and output time waveforms in whatever form they may be, and to calculate H using S_x , S_y and the Fourier transform.

For the general measurement case, the input $x(t)$ is not sinusoidal and will often be chosen to be random noise, especially since it has several advantages when used as a stimulus for measuring structural transfer functions. However, it is not generally useful to measure the linear spectrum of this type of signal because it cannot be smoothed by averaging; therefore we typically resort to the power spectrum.

The power spectrum of the system input is defined and computed as:

$$\begin{aligned} G_{xx} &= \text{Power spectrum of the input } x(t) \\ &= S_x S_x^* \end{aligned} \tag{32}$$

where

$$S_x^* = \text{Complex conjugate of } S_x.$$

and

$$\begin{aligned} G_{yy} &= \text{Power spectrum of the output } y(t) \\ &= S_y S_y^* \end{aligned} \tag{33}$$

where

$$S_y^* = \text{Complex conjugate of } S_y.$$

The cross power spectrum between the input and the output is denoted by G_{yx} and defined as,

$$G_{yx} = S_y S_x^* \tag{34}$$

Returning to equation (31), we can multiply the numerator and denominator by S_x^* . This shows that the transfer

function can be expressed as the ratio of the cross power spectrum to the input auto power spectrum.

$$H = \frac{S_y}{S_x} \cdot \frac{S_x^*}{S_x^*} = \frac{G_{yx}}{G_{xx}} \tag{35}$$

There are three important reasons for defining the system transfer function in this way. First, this technique measures magnitude and phase since the cross power spectrum contains phase information. Second, this formulation is not limited to sinusoids, but may in fact be used for any arbitrary waveform that is Fourier transformable (as most physically realizable time functions are). Finally, averaging can be applied to the measurement. This alone is an important consideration because of the large variance in the transfer function estimate when only one measurement is used. So, in general,

$$H(f) = \frac{\overline{G_{yx}}}{\overline{G_{xx}}} \tag{36}$$

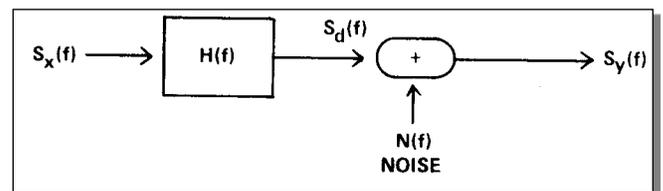
where $\overline{G_{yx}}$ denotes the ensemble average of the cross power spectrum and $\overline{G_{xx}}$ represents the ensemble average of the input auto power spectrum.

As an added note, the impulse response $h(t)$ of a linear system is merely the inverse transform of the system transfer function,

$$h(t) = F^{-1} \left\{ \frac{\overline{G_{yx}}}{\overline{G_{xx}}} \right\} \tag{37}$$

Reducing Measurement Noise

The importance of averaging becomes much more evident if the transfer function model shown above is expanded to depict the "real-world" measurement situation. One of the major characteristics of any modal testing system is that extraneous noise from a variety of sources is always measured along with the desired excitation and response signals. This case for transfer function measurements is shown below.



where:

$S_x(f)$ = Lineal Fourier spectrum of the measured input signal

$S_d(f)$ = Linear Fourier spectrum of the desired response measurement

$S_y(f)$ = Linear Fourier spectrum of the measured response

$N(f)$ = Linear Fourier spectrum of the noise

$H(f)$ = System transfer function

Since we are interested in identifying modal parameters from measured transfer functions, the variance on the parameter estimates is reduced in proportion to the amount of noise reduction in the measurements. The digital Fourier analyzer has two inherent advantages over other types of analyzers in reduction of measurement noise; namely, ensemble averaging, and a second technique commonly referred to as post data smoothing which may be applied after the measurements are made.

Without repeating the mathematics for the general model of a transfer function measurement in the presence of noise, it is easy to show that the transfer function is more accurately written as:

$$H = \frac{\overline{G_{yx}}}{\overline{G_{xx}}} - \frac{\overline{G_{nx}}}{\overline{G_{xx}}} \quad (38)$$

where the frequency dependence notation has been dropped and,

$\overline{G_{yx}}$ = Ensemble average of the cross power spectrum between input and output

$\overline{G_{xx}}$ = Ensemble average of the input power spectrum

$\overline{G_{nx}}$ = Ensemble average of the cross power spectrum between the noise and the input

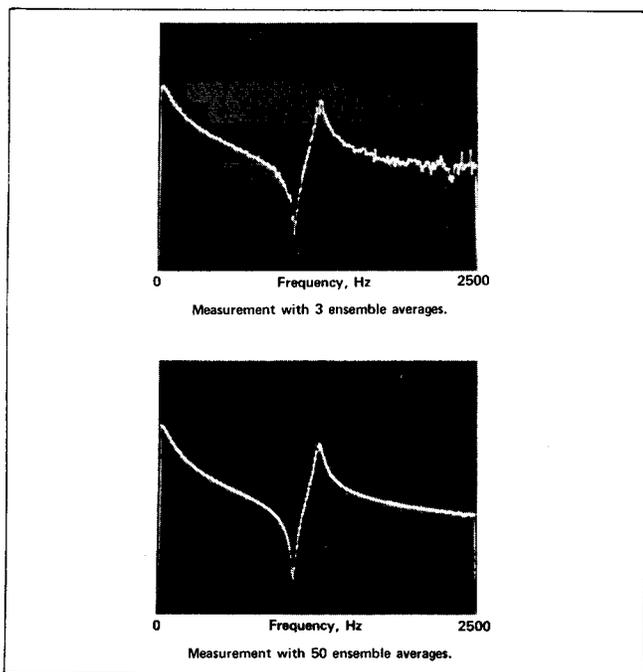


Figure 11—Effect of averaging on transfer function measurements.

This form assumes that the noise has a zero mean value and is incoherent with the measured input signal. Now, as the number of ensemble averages becomes larger, the noise term $\overline{G_{nx}}$ becomes smaller and the ratio $\overline{G_{yx}}/\overline{G_{xx}}$ more accurately estimates the true transfer function. Figure 11 shows the effect of averaging on a typical transfer function measurement.

The Coherence Function

To determine the quality of the transfer function, it is not sufficient to know only the relationship between input and output. The question is whether the system output is totally caused by the system input. Noise and/or non-linear effects can cause large outputs at various frequencies, thus introducing errors in estimating the transfer function. The influence of noise and/or non-linearities, and thus the degree of noise contamination in the transfer function is measured by calculating the coherence function, denoted by γ^2 , where

$$\gamma^2 = \frac{\text{Response power caused by applied input}}{\text{Measured response power}} \quad (39)$$

The coherence function is easily calculated on a digital Fourier analyzer when transfer functions are being measured. It is calculated as:

$$\gamma^2 = \frac{|\overline{G_{yx}}|^2}{\overline{G_{xx}} \overline{G_{yy}}} \quad \text{where } 0 \leq \gamma^2 \leq 1 \quad (40)$$

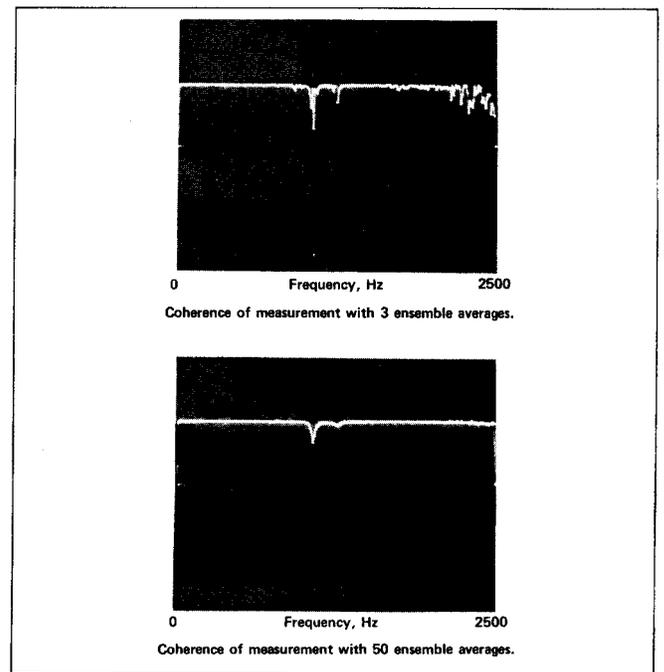


Figure 12—Effect of ensemble averaging on the coherence for the transfer function shown in Figure 11.

If the coherence is equal to 1 at any specific frequency, the system is said to have perfect causality at that frequency. In other words, the measured response power is caused totally by the measured input power (or by sources which are coherent with the measured input power). A coherence value less than 1 at a given frequency indicates that the measured response power is greater than that due to the measured input because some extraneous noise is also contributing to the output power.

When the coherence is zero, the output is caused totally by sources other than the measured input. In general terms, the coherence is a measure of the degree of noise contamination in a measurement. Thus, with more averaging, the estimate of coherence contains less variance, therefore giving a better estimate of the noise energy in a measured signal. This is illustrated in Figure 12.

Since the coherence function indicates the degree of causality in a transfer function it has two very important uses:

- 1) It can be used qualitatively to determine how much averaging is required to reduce measurement noise.
- 2) It can serve as a monitor on the quality of the transfer function measurements.

The transfer functions associated with most mechanical systems are so complex in nature that it is virtually impossible to judge their validity solely by inspection. In one case familiar to the author, a spacecraft was being excited with random noise in order to obtain structural transfer functions for modal parameter identification. The transfer and coherence functions were monitored for each measurement. Then, between two measurements the coherence function became noticeably different from unity. After rechecking all instrumentation, it was discovered that a random vibration test being conducted in a separate part of the same building was providing incoherent excitation via structural (building) coupling, even through a seismic isolation mass. This extraneous source was increasing the variance on the measurement but would probably not have been discovered without use of the coherence function.

Summary

In Part I, we have introduced the structural dynamic model for elastic structures and the concept of a mode of vibration in the Laplace domain. This means of representing modes of vibration is very useful because we are interested in identifying the modal parameters from measured frequency response functions. Lastly, the procedure for calculating transfer and coherence functions in a digital Fourier analyzer were discussed.

In Part II, we will discuss various techniques for accurately measuring structural transfer functions. Because modal parameter identification algorithms work on actual measured data, we are interested in making the best measurements possible, thus increasing the accuracy of our parameter estimates. Techniques for exciting struc-

tures with various forms of excitation will be discussed. Also, we will discuss methods for arbitrarily increasing the available frequency resolution via band selectable Fourier analysis-the so-called zoom transform.

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