Determination of Modal Sensitivity Functions for Location of Structural Faults

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Abstract

It is well known that the modal properties of a structure are directly related to its mass, stiffness, and damping properties. In fact, the modal parameters (eigenvalues and eigenvectors), are solutions to the differential equations of motion, which are written in terms of the mass, stiffness, and damping.

This paper focuses on the determination of the functional relationship between variations in the mass, stiffness, and damping, and variations in the modal properties of structures. For small changes, this "sensitivity function" reduces to a very simple function of variations in the modal frequencies and damping only. This makes it possible to *detect, locate, and quantify* structural faults by monitoring *frequency and damping only*. This finding was previously reported by Stubbs et.al. [4], [5], [6].

In this paper the complete sensitivity functions for mass, stiffness, and damping changes are derived, and the validity of the stiffness sensitivity for small changes is verified. It is also pointed out that the Structural Dynamics Modification (SDM) technique can be used to determine the additional terms for the complete sensitivity formulas.

Nomenclature

n = number of degrees-of-freedom (DOFs) of the structural dynamic model

m, modes = number of modes

$\left[M\right]$ = $\left(n \text{ by } n\right)$ mass matrix

$\left[C\right]$ = $\left(n \text{ by } n\right)$ damping matrix

$\left[K\right]$ = $\left(n \text{ by } n\right)$ stiffness matrix

$\left\{x''(t)\right\}$ = acceleration n-vector

$\left\{x'(t)\right\}$ = velocity n-vector

$\left\{x(t)\right\}$ = displacement n-vector

$\left\{f(t)\right\}$ = external force n-vector

$\left[-m-\right]$ = $\left(m \text{ by } m\right)$ modal mass matrix

$\left[-c-\right]$ = $\left(m \text{ by } m\right)$ modal damping matrix

$\left[-k-\right]$ = $\left(m \text{ by } m\right)$ modal stiffness matrix

w_k = damped natural frequency for mode k

d_k = damping coefficient for mode k

W_k = undamped natural frequency for mode k

$\left[U\right]$ = $\left(n \text{ by } m\right)$ mode shape matrix

Introduction

The modes of vibration of a structure are strongly influenced by slight changes in its physical properties or its boundary conditions. This fact is also self evident if one considers the mathematical definition of modes as the eigenvalue solutions of the differential equations of motion for a vibrating structure. These equations result from a straightforward application of Newton's Second Law to the structure, and define a *force balance* between the inertial (mass), dissipative (damping), and restoring (stiffness) forces within the structure, and the externally applied forces. These equations are also related, via the Fourier transform, to the Frequency Response Function (FRF) form of the dynamic model, which can also be represented in terms of the modal parameters of the structure. This parametric form of the FRF matrix in terms of modal parameters is the foundation upon which all modern day modal testing is done. Finally, the Impulse Responses of a structure comprise a third, completely equivalent form of the dynamic model, and they too can be represented parametrically in terms of modes of vibration. Experimentally determined impulse responses are also used in modern day modal test systems to identify modal parameters. Figure 1 illustrates this interdependency between the physical properties, (distributed mass, stiffness, and damping), the FRFs, the impulse responses, and the modal properties of a structure. It is clear, then, that changes in the physical properties or boundary conditions, (both of which affect the mass, stiffness, and damping properties), will cause changes in the measured FRFs or impulse response functions, and also will change the measured modal properties.

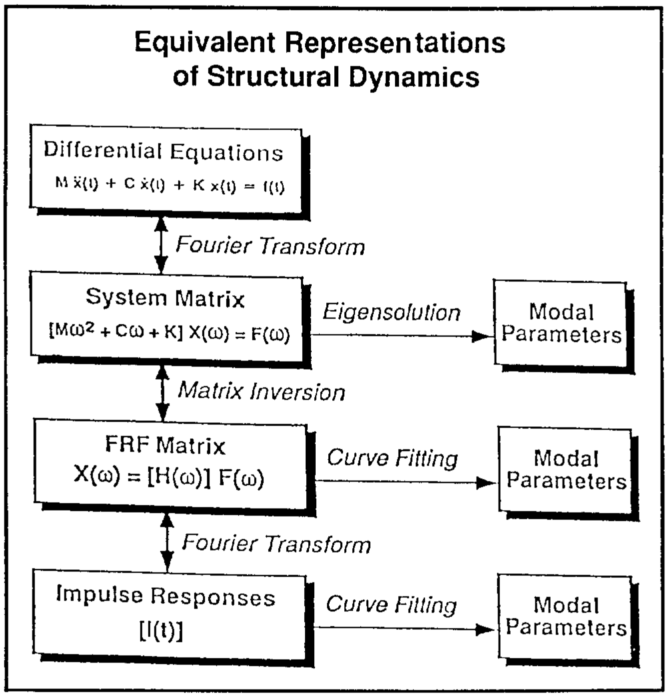

Figure 1

In previous papers, [1], [2], [3], we investigated ways in which the current modal testing technology could be used for detecting and locating structural faults. A structural fault could be any one of the occurrences listed in Figure 2. Using modern day digital testing equipment, a structure can be excited in a wide variety of ways, and high resolution, noise free, linear estimates of its FRFs can be obtained. Modal parameter identification methods can then be applied to the FRF measurements to very accurately determine the modal properties of the structure. The technology exists for detecting **millihertz changes** in modal frequencies and damping, and also for obtaining changes in mode shapes, if necessary.

In the following development, we expand upon the work of Stubbs, et.al. [4], [5], [6]. They used the orthogonality condition for classically damped (or lightly damped) structures, and derived relationships between changes in the mass, stiffness, and damping matrices and changes in the modal parameters. The resulting equations can be used directly to locate and quantify damage (physical change) on a structure, if the undamaged mass, stiffness, and damping plus measured changes in the modal properties are known.

A key finding of theirs, however, and one which we will verify here also, is that if the modal shapes don't change appreciably, (this usually holds for “small” changes in the physical properties), then **damage can be located and quantified by using only changes in the modal frequencies and damping, plus the mode shapes of the undamaged structure.** This offers a decided advantage from an implementation standpoint since modal frequency and damping can be easily measured at practically any point on a structure.

What is a Structural Fault?

- **Failure of the Structural Material due to Fatigue.** For example, cracking, breaking, delamination.
- **Loosening of Assembled Parts.** For example, loose bolts, rivets, glued joints, or wear-out of parts.
- **Manufacturing Defects.** For example, flaws voids or thin spots due to casting, molding, or forming operations. Improper assembly of parts.

Figure 2

Theoretical Background

Modes of vibration are commonly defined as solutions to the following differential equations:

$$[M]\{x''(t)\}+[C]\{x'(t)\}+[K]\{x(t)\}=\{f(t)\} \qquad (1)$$

The modal properties are actually solutions to the homogeneous equations (i.e. where $\{f(t)\}=\{0\}$), and are found by a straightforward eigensolution process. The coefficient matrices ($[M]$, $[C]$, and $[K]$) are usually assumed to be real valued and symmetric, and without any further assumptions, complex conjugate pairs of eigenvalues and corresponding eigenvectors can be found from these equations. These constitute the so-called complex modes of the structure. However, if the **damping term is assumed to be negligible compared to the mass and stiffness terms** in equation (1), the eigenvalues and eigenvectors exhibit a very strong orthogonality property, which will be exploited here. The above assumption can be applied to the majority of real world structures, and certainly to those which vibrate freely. Another way of stating it is that the **damping forces are assumed to be negligible compared to the inertial and restoring forces of the structure.** Such structures are said to be classically damped, or simply lightly damped.

The orthogonality property of the modes “almost” simultaneously diagonalizes the mass, stiffness, and damping matrices, and therefore “almost” uncouples the equations of motion. The term “almost” is used because strict diagonalization only occurs if there is no damping $([C]=[0])$, or if the damping matrix is *proportional* to the mass and stiffness matrices, a difficult assumption to verify with real structures. Nevertheless, when damping is negligible, the following orthogonality conditions can be applied to the unmodified structure (structure with no fault):

$$[U]^t[M][U]=[-m-] \qquad (2)$$

$$[U]^t[C][U]=[-c-] \quad (3)$$

$$[U]^t[K][U]=[-k-] \quad (4)$$

Two other relationships which result from orthogonality are:

$$W_{0k}^{\,2}=k_k/m_k \qquad k=1\text{...modes} \quad (5)$$

$$2d_k=c_k/m_k \qquad k=1\text{...modes} \quad (6)$$

where: $W_{0k}^{\,2}=w_{0k}^{\,2}+d_{0k}^{\,2}$

and: $w_{0k}^{\,2},\ d_{0k}^{\,2}$ = frequency and damping of the unmodified structure

Equations (2) through (6) can also be written for the modified structure (structure with a fault):

$$[U+dU]^t[M+dM][U+dU]=[-m+dm-] \quad (7)$$

$$[U+dU]^t[C+dc][U+dU]=[-c+dc-] \quad (8)$$

$$[U+dU]^t[K+dK][U+dU]=[-k+dk-] \quad (9)$$

$$W_{1k}^{\,2}=(k_k+dk_k)/(m_k+dm_k) \qquad k=1\text{...modes} \quad (10)$$

where: $W_{1k}^{\,2}=w_{1k}^{\,2}+d_{1k}^{\,2}$

$$2d_{1k}=(c_k+dc_k)/(m_k+dm_k) \qquad k=1\text{...modes} \quad (11)$$

Notice that each of physical and modal properties of the modified structure is written in terms of the same property of the unmodified structure, plus an additive change "d" term. Therefore, equations (7) through (11) can be expanded and the unmodified conditions *subtracted* from them to yield a new set of formulas that relate changes in the mass, stiffness and damping matrices to changes in the modal properties:

$$\begin{aligned}&[U+dU]^t[dM][U+dU]+2[dU]^t[M][U]\\&\qquad+[dU]^t[M][dU]=[-dm-]\end{aligned} \quad (12)$$

$$\begin{aligned}&[U+dU]^t[dC][U+dU]+2[dU]^t[C][U]\\&\qquad+[dU]^t[C][dU]=[-dc-]\end{aligned} \quad (13)$$

$$\begin{aligned}&[U+dU]^t[dK][U+dU]+2[dU]^t[K][U]\\&\qquad+[dU]^t[K][dU]=[-dk-]\end{aligned} \quad (14)$$

$$m_k\left(W_{1k}^{\,2}-W_{0k}^{\,2}\right)+dm_kW_{1k}^{\,2}=dk_k \quad (15)$$

$$2m_k\left(d_{1k}-d_{0k}\right)+2dm_kd_{1k}=dc_k \quad (16)$$

Stiffness Changes

Probably the most sought after cause of a structural fault is a reduction in local stiffness, which might be caused by the formation of a crack, delamination, or a loose fastener. The above equations can be combined to yield a single relationship between changes in the stiffness matrix $[dK]$ and changes in the modal parameters.

Mode mass (m_k) is simply a scaling constant and therefore, can be arbitrarily set to any value. We can always scale the mode shapes to **unity modal masses,** so that m_k = 1 and (m_k+dm_k) = 1, which also implies that dm_k = 0 for all modes (k).

Combining equations (14) and (15) and using mode shapes scaled to unit modal masses gives the following **stiffness sensitivity equation:**

$$\begin{aligned}&\{U_k+dU_k\}^t[dK]\{U_k+dU_k\}+\{dU_k\}^t[K]\{U_k\}\\&\quad+\{dU_k\}^t[K]\{dU_k\}=w_{1k}^{\,2}-w_{0k}^{\,2} \qquad k=1\text{...modes}\end{aligned} \quad (17)$$

This formula expresses changes in the stiffness matrix $[dK]$ as functions of the unmodified stiffness $[K]$ and changes in the modal properties, i.e. change in the mode shape $\{dU_k\}$, plus changes in the modal frequency $\left(w_{1k}^{\,2}-w_{0k}^{\,2}\right)$. Stiffness changes don't affect modal damping. Notice that an equation (17) can be written for each mode (k), which means that a *set of (m) equations can be solved for up to (m) stiffness changes at a time*. Hence, when the number of unknown local stiffness changes exceeds the number of modes for which we have measured changes, some sort of a searching algorithm will be required. This issue is considered in the numerical example to follow.

"Small" changes: If it can be further assumed that the fault is slight enough so that the mode shapes don't change substantially, (i.e. $\{dU_k\}$ = $\{0\}$), then the stiffness sensitivity equation is greatly simplified:

$$\{U_k\}^t[dK]\{U_k\}=w_{1k}^{\,2}-w_{0k}^{\,2} \qquad k=1\text{...modes} \quad (18)$$

Local Stiffness Changes

A stiffness change between two degrees-of-freedom, say DOF_i and DOF_j, changes the stiffness matrix in the following manner:

$$[dK] = \begin{array}{c c} & \begin{array}{cc} \mathrm{DOF}_i & \mathrm{DOF}_j \end{array} \\ \begin{array}{c} \mathrm{DOF}_i \\ \mathrm{DOF}_j \end{array} & \begin{bmatrix} dk_{ij} & -dk_{ij} \\ -dk_{ij} & dk_{ij} \end{bmatrix} \end{array}$$

Equation (18) can therefore be rewritten in terms of local stiffness changes as:

$$\underset{i}{\mathrm{sum}}\,\underset{j}{\mathrm{sum}}\left(u_{ik}^2 + u_{jk}^2 - 2u_{ik}u_{jk}\right)dk_{ij} = w_{1k}^{\ 2} - 1_{0k}^{\ 2}$$
$$k = 1\ldots\text{modes} \quad (19)$$

This formula only requires the mode shapes for the unmodified structure plus changes in the frequency of the modes. The modal parameters of the unmodified structure can be obtained either by modal testing or finite element analysis. A fault which causes a local stiffness change can then be detected, located, and quantified by simply tracking the modal frequencies of the structure, and using equation (19).

Mass Changes

Equations (12), (14), and (15) can be combined in the

same manner as above to yield a **mass sensitivity equation:**

$$\{U_k + dU_k\}^t [dM]\{U_k + dU_k\} + 2\{dU_k\}^t [M]\{U_k\} + \{dU_k\}^t [M]\{dU_k\} = \left.\begin{pmatrix} 2\{dU_k\}^t [K]\{U_k\} \\ + \{dU_k\}^t [K]\{dU_k\} \\ -\left(W_{1k}^{\ 2} - W_{0k}^{\ 2}\right) \end{pmatrix}\right/ W_{1k}^{\ 2}$$
$$k = 1\ldots\text{modes} \quad (20)$$

This formula requires both the mass $[M]$ and stiffness $[K]$ of the unmodified structure, plus changes in the modal parameters due to the fault. Again, as with the stiffness, for “small” changes which don't affect the mode shapes substantially, the mass sensitivity equation becomes greatly simplified:

$$\{U_k\}^t [dM]\{U_k\} = -\left(W_{1k}^{\ 2} - W_{0k}^{\ 2}\right)\big/ W_{1k}^{\ 2}$$
$$k = 1\ldots\text{modes} \quad (21)$$

Damping Changes

Equations (13) and (17) can be combined, together with unit modal mass scaling of the mode shapes, to yield a **damping sensitivity equation:**

$$\{U_k + dU_k\}^t [dC]\{U_k + dU_k\} + 2\{dU_k\}^t [C]\{U_k\} + \{dU_k\}^t [C]\{dU_k\} = 2\left(d_{1k} - d_{0k}\right)$$
$$k = 1\ldots\text{modes} \quad (22)$$

And, for “small” changes which don't affect the mode shapes, this equation simplifies to:

$$\{U_k\}^t [dC]\{U_k\} = 2\left(d_{1k} - d_{0k}\right) \qquad k = 1\ldots\text{modes} \quad (23)$$

A 3-DOF Example

The validity of stiffness sensitivity, equation (19), will be tested on the 3-DOF vibratory structure shown in Figure 3. The mass, stiffness, and damping, as well as the modal properties of the structure are also given in Figure 3.

Fourteen different stiffness changes were made to the 3-DOF structure to simulate different fault conditions. These are shown in Figure 4. These changes involved various combinations of local stiffness changes between DOFs 1x, 2x, and 3x. (Since all of the DOFs are in the x-direction, only the point numbers are used as subscripts). The modal frequency shifts caused by the simulated stiffness changes are given in Figure 5. These were obtained by solving for the modes of the modified structure with each stiffness change using the Structural Dynamics Modification (SDM) technique.

Since modal data for three modes is available, equation (19) can be written for all three modes and the resulting set of equations solved for three stiffness changes at a time. Those results are shown in Figure 6. Notice that 5% and 10% changes of stiffness are all satisfactorily predicted, (cases 1, 2, 4, 5, 7, 8, & 10). The 15% to 25% changes (cases 3, 6, 9, 11, 12 & 13) were also *correctly located*, although with a larger amount of numerical error. Case 14, with 30% overall reduction in stiffness, also shows a significant degree of error.

As a second test, we simulated a more realistic situation where *either all the modes of the structure have not been measured, or else the number of potential local stiffness changes exceeds the number of modes measured*. In this test, we used sensitivity equations for only the first two modes of the 3-DOF structure. This meant that we could only solve for a single stiffness change, or two changes at a time. The two-at-a-time solutions are shown in Figure 7. Notice that there are a number of unrealistic (positive) changes, which are double underlined.

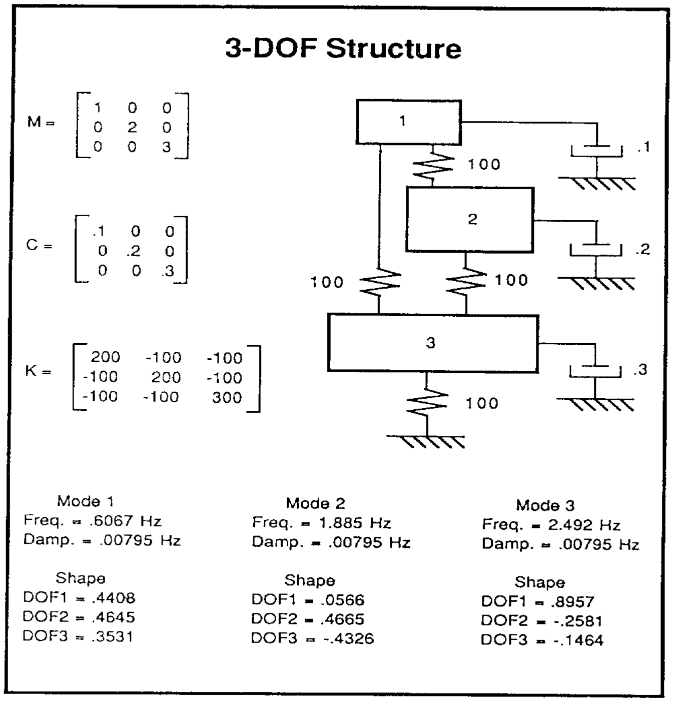

Figure 3

Whenever an unrealistic change occurred, that local stiffness was eliminated from further consideration as a possible fault location. The summary of the search through the data in Figure 7 is shown in Figure 8. Notice that the single changes in cases 1 through 9 were all correctly located, and that cases 10 through 14 encountered some difficulty. For example, in case 10 the correct answer is found when only dk_{12} and dk_{13} are used as unknowns, but when dk_{12} and dk_{23} are used, dk_{12} is positive and is therefore rejected as a possible solution candidate.

Figure 4. Simulated Fault Cases

Stiffness change	case 1	case 2	case 3	case 4	case 5	case 6	case 7
dk_{12}	-5	-10	-25	0	0	0	0
dk_{13}	0	0	0	-5	-10	-25	0
dk_{23}	0	0	0	0	0	0	-5

Stiffness change	case 8	case 9	case 10	case 11	case 12	case 13	case 14
dk_{12}	0	0	-2.5	-10	0	-5	-10
dk_{13}	0	0	-2.5	-10	-10	-5	-10
dk_{23}	-10	-25	0	0	-10	-5	-10

Figure 5. Modal Frequency Shifts $\left(f_1^2 - f_0^2\right)$

Mode	case 1	case 2	case 3	case 4	case 5	case 6	case 7
1	-0.0029	-0.0060	-0.0170	-0.0398	-0.0824	-0.2305	-0.0641
2	-0.8963	-1.9189	-6.0000	-1.2613	-2.6614	-7.8682	-4.0426
3	-6.6011	-13.0753	-31.4833	-5.3656	-10.5895	-25.2347	-0.0602

Mode	case 8	case 9	case 10	case 11	case 12	case 13	case 14
1	-0.1326	-0.3699	-0.0209	-0.0868	-0.2220	-0.1085	-0.2284
2	-8.0849	-20.2024	-1.0185	-4.0733	-10.6208	-6.0754	-12.1424
3	-0.1159	-0.2612	-6.0441	-24.1734	-10.8241	-12.1495	-24.2961

Figure 6. Calculated Stiffness Changes

Stiffness change	case 1	case 2	case 3	case 4	case 5	case 6	case 7
dk_{12}	-5.0830	-10.3416	-27.5628	0.0816	0.3449	2.3692	0.2931
dk_{13}	0.1547	0.6441	4.8380	-5.0388	-10.1677	-26.1074	-0.3578
dk_{23}	-0.0984	-0.4157	-3.1280	-0.0849	-0.3525	-2.4929	-4.9553

Stiffness change	case 8	case 9	case 10	case 11	case 12	case 13	case 14
dk_{12}	1.2341	8.6375	-2.4651	-9.3826	2.7835	-4.2458	-6.8347
dk_{13}	-1.5068	-10.5567	-2.5427	-10.7547	-13.2646	-5.9247	-13.8808
dk_{23}	-9.8106	-23.6571	0.0054	0.0960	-9.7872	-4.8781	-9.4885

Conclusions

We have verified by example that the orthogonality conditions for classically damped structures can be used to accurately locate and quantify structural faults, by simply using changes in measured modal frequencies. This is indeed a powerful result, which was previously pointed out by Stubbs, et.al. [4], [5], [6]. We derived separate sensitivity equations for mass, stiffness, and damping changes, and we showed that unity modal mass scaling of the mode shapes simplifies these formulas.

We found from the numerical example that *changes of 10% or less in stiffness* could be accurately predicted using the "small" change version of the formulas. This form assumes that the mode shapes don't change significantly. When a sufficient number of modes are monitored, even changes as large as 25% can still be correctly located. But with a reduced set of modes, which is more applicable to real world problems, the small change equations still did an adequate job of locating changes of 10% or less.

When the small change assumption cannot be made, a substantially greater amount of data is required to use the sensitivity equations. Not only are the mass, stiffness, and damping matrices of the unmodified structure needed, but the mode shapes of the modified structure would also have to be measured as well. For specific applications, however, this additional work may be warranted in order to obtain the increased accuracy.

Finally, a set of modal parameters for an unmodified (undamaged) structure and the SDM technique could be used to determine all of the unknown mass, stiffness, and damping terms in the sensitivity equations (17), (20), and (22). Or, from a different perspective, these sensitivity equations together with the use of the SDM technique provide another way of estimating the mass, stiffness and damping matrices of a structure from measured modal data.

References

[1] Wolff, T. and Richardson, M. **"Fault Detection in Structures from Changes in Their Modal Parameters"** Proceedings of the 7th International Modal Analysis Conference, Las Vegas, Nevada, SEM, Bethel, CT.

[2] Richardson, M. and Mannan, M.A. **"Detection and Location of Structural Cracks Using FRF Measurements"** 8th International Modal Analysis Conference, Kissimmee Florida, Jan 29 - Feb 1, 1990, SEM, Bethel, CT.

[3] Richardson, M. and Mannan, M.A. **"Using Measured Modal Parameters and the Stiffness Matrix to Detect and Locate Structural Faults"** ICSTAD Proceedings, Jul 29 - Aug 3, 1990, Bangalore, India.

[4] Stubbs, N., Broome, T.H., and Osegueda, R. **"Non-Destructive Construction Error Detection in Large Space Structures"** AIAA Journal, Vol. 28, No.1,1990.

[5] Stubbs, N. and Osegueda, R. **"Global Non -Destructive Damage Evaluation in Solids",** International Journal of Analytical and Experimental Modal Analysis, Vol. 5, No. 2, April, 1990, pp.67-79.

[6] Stubbs, N. and Osegueda, R. **"Global Damage Detection in Solids-Experimental Verification",** International Journal of Analytical and Experimental Modal Analysis, Vol. 5, No. 2, April, 1990, pp.81-97.

Figure 7. Calculated Stiffness Changes Using Only Two Modes

(unknowns dk_{12} & dk_{13})

Stiffness change	case 1	case 2	case 3	case 4	case 5	case 6	case 7
dk_{12}	-5.3591	- 11.508	-36.9389	-0.1565	0.6441	-4.6251	-13.6099
dk_{13}	0.0161	0.0586	0.4325	-5.1583	-10.6644	-29.6186	-7.3371

Stiffness change	case 8	case 9	case 10	case 11	case 12	case 13	case 14
dk_{12}	-26.2913	-57.7365	-2.4501	-9.1134	-24.6761	-4.2458	-6.8347
dk_{13}	-15.3244	-43.8759	-2.5351	-10.6195	-27.0490	-5.9247	-13.8808

(unknowns dk_{12} & dk_{23})

Stiffness change	case 1	case 2	case 3	case 4	case 5	case 6	case 7
dk_{12}	-5.3912	-11.6247	-37.2005	10.1192	20.5997	54.3769	1.0060
dk_{23}	0.0114	0.0416	03071	-3.6625	-7.5717	-21.0296	-5.2094

Stiffness change	case 8	case 9	case 10	case 11	case 12	case 13	case 14
dk_{12}	4.2359	29.6671	2.6001	12.0414	29.2073	7.5566	20.8167
dk_{23}	-10.8805	-31.1525	-1.8000	-7.5400	-19.2052	-9.0847	-19.3440

(unknowns dk_{13} & dk_{23})

Stiffness change	case 1	case 2	case 3	case 4	case 5	case 6	case 7
dk_{13}	2.7063	5.8355	18.6744	-5.0798	-10.3409	-27.2968	-0.5050
dk_{23}	-1.9101	-4.1017	-12.9520	-0.0558	-0.2296	-1.6485	-4.8509

Stiffness change	case 8	case 9	case 10	case 1 l	case 12	case 13	case 14
dk_{13}	-2.1264	-14.8926	-1.3052	-6.0447	-14.6618	-3.7933	-10.4498
dk_{23}	-9.3708	-20.5785	-0.8733	-3.2482	-8.7951	-6.3914	-11.9245

Figure 8. Summary of Search Through Two Mode Results
(all positive dk s set to zero)

Stiffness change	case 1	case 2	case 3	case 4	case 5	case 6	case 7
dk_{12}	-5.3361	-11.4246	-35.7223	0	0	0	0
dk_{13}	0	0	0	-5.0798	-10.3409	-27.2968	-0.5050
dk_{23}	0	0	0	-0.0558	-0 2296	-1.6485	-4.8509

Stiffness change	case 8	case 9	case 10*	case 11 *	case 12	case 13*	case 14*
dk_{12}	0	0	0	0	0	0	0
dk_{13}	-2.1264	-14.8926	-1.3052	-6.0447	-14.6618	-3.7933	-10.4498
dk_{23}	-9.3708	-20.5785	0.8733	-3.2482	-8.7951	-6.3914	-11.9245