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Curve Fitting Analytical Mode Shapes to Experimental Data

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ABSTRACT

In this paper, we employ the fact that all experimental vibration data, whether in the form of a set of FRFs or a set of output-only spectra, is a ***summation of resonance curves***, each curve due to a mode of vibration. We also use this superposition property of modes to calculate a ***modal participation matrix***, a measure of the participation of each mode in the experimental vibration data

First we show how this ***superposition*** property can be used to curve fit a set of FEA mode shapes to EMA mode shapes or ODS's. The modal participation matrix is calculated as a ***least-squared-error solution***, so any number of FEA mode shapes can be curve fit to any number of EMA mode shapes or ODS's. Next we show how an expanded and enhanced set of FRFs, Cross spectra or ODS FRFs is obtained by curve fitting FEA mode shapes to experimental data.

This approach in an alternative to FEA Model Updating, where an FEA model is modified so that its modes more closely correlate with experimental data. By curve fitting FEA shapes to experimental data, an extending and enhanced dynamic model is obtained which is more suitable for machinery & structural health monitoring, and for troubleshooting noise & vibration problems using SDM and MIMO methods.

KEY WORDS

Frequency Response Function (FRF)
Auto power spectrum (APS)
Cross power spectrum (CPS)
ODSFRF
Operating Deflection Shape (ODS)
Finite Element Analysis (FEA)
Experimental Modal Analysis (EMA)
FEA Mode Shapes
EMA Mode Shapes
Modal Assurance Criterion (MAC)
Shape Difference Indicator (SDI)
Structural Dynamics Modification (SDM)
Multi-Input Multi-Output (MIMO) Modeling & Simulation

INTRODUCTION

When resonances are excited by dynamic forces in a machine, or in a mechanical or civil structure, response levels can ***far exceed*** deformation levels due to static loads. Moreover, high levels of ***resonance-assisted dynamic response*** can cause rapid and unexpected failures, or over long periods of time, structural fatigue and material failure can often occur.

A mode of vibration is a mathematical representation of a structural resonance. Each mode is defined by three distinct numerical parts; a natural frequency, a damping or decay constant, and a mode shape. A mode shape represents the ***"standing wave deformation"*** of the structure at the natural frequency of a resonance. This standing wave behavior is caused when energy becomes trapped within the material boundaries of the structure and cannot readily escape.

Modal Participation

When the dynamic response of a structure is expressed in terms of modal parameters, every solution is a ***summation of contributions*** from all of the modes. Another way of expressing this superposition property is that ***all modes participate*** in or ***contribute*** to the dynamic response of a structure when it is excited by applied forces. Ideally, all structures have an ***infinite number*** of modes, but in a practical sense only a ***few low frequency modes participate significantly*** in their response.

If a structure's overall dynamic response is represented by time waveforms, these waveforms can be decomposed into a ***summation of modal contributions***. Likewise, if the dynamic response is represented by a set of frequency functions or spectra, the overall response can also be decomposed into a ***summation of resonance curves***.

FEA Modes

FEA modes are solutions to a set of time domain equations of motion. The equations are a statement of Newton's second law [3]-[5], a force balance between internal and external forces. This force balance is written as a set of differential equations,

$$[\mathbf{M}]\{\ddot{\mathbf{x}}(\mathbf{t})\}+[\mathbf{C}]\{\dot{\mathbf{x}}(\mathbf{t})\}+[\mathbf{K}]\{\mathbf{x}(\mathbf{t})\}=\{\mathbf{f}(\mathbf{t})\} \qquad (1)$$

where,

$[\mathbf{M}]=$ **(n by n)** mass matrix
$[\mathbf{C}]=$ **(n by n)** damping matrix
$[\mathbf{K}]=$ **(n by n)** stiffness matrix
$\{\ddot{\mathbf{x}}(\mathbf{t})\}=$ Accelerations (**n**-vector)
$\{\dot{\mathbf{x}}(\mathbf{t})\}=$ Velocities (**n**-vector)
$\{\mathbf{x}(\mathbf{t})\}=$ Displacements (**n**-vector)
$\{\mathbf{f}(\mathbf{t})\}=$ Externally applied forces (**n**-vector)

This set of differential equations describes the dynamics between ***n-discrete degrees-of-freedom (DOFs)*** of the structure. Equations can be created for as many DOFs of a structure as necessary to adequately describe its dynamic behavior.

Finite element analysis (FEA) is used to create the coefficient matrices of the differential equations (1). The mass & stiffness matrices can be generated in a straightforward manner but the damping matrix cannot, hence it is usually left out of the FEA model. The equations are then solved for the analytical FEA mode shapes and their corresponding natural frequencies.

FEA modes are a mathematical ***eigen-solution*** to the homogeneous form of equations (1), where the right-hand side is zero. Each natural frequency is an ***eigenvalue***, and each mode shape is an ***eigenvector.***

EMA Modes

EMA mode shapes are obtained by ***curve fitting*** a set of experimentally derived FRFs [3]-[5]. FRF-based curve fitting is a numerical process by which an analytical parametric model with unknown modal parameters in it is matched to experimental FRF data over a band of frequencies. Equation (3) is an expression of the analytical FRF model.

In the frequency domain, the equations of motion are written as algebraic equations, in a form called a ***MIMO model*** or ***transfer function model***. Like equation (1), this model also describes the ***dynamics between n-DOFs*** of the structure. It contains transfer functions between all combinations of DOF pairs,

$$\{X(s)\} = [H(s)]\{F(s)\} \qquad (2)$$

where,

$s =$ Laplace variable (complex frequency)

$[H(s)] =$ (**n by n**) matrix of transfer functions

$\{X(s)\} =$ Laplace transform of displacements (**n**-vector)

$\{F(s)\} =$ Laplace transform of external forces (**n**-vector)

These equations can be created between as many DOF pairs of the structure as necessary to adequately describe its dynamic behavior.

When a transfer function is represented analytically as the partial fraction expansion shown in equations (3) & (4) below, it is clear that its value at any frequency is a ***summation of resonance curves***, one for each mode of vibration.

$$[H(s)] = \sum_{k=1}^{m} \frac{[r_k]}{2j(s - p_k)} - \frac{[r_k^*]}{2j(s - p_k^*)} \qquad (3)$$

or,

$$[H(s)] = \sum_{k=1}^{m} \frac{A_k\{u_k\}\{u_k\}^t}{2j(s - p_k)} - \frac{A_k^*\{u_k^*\}\{u_k^*\}^t}{2j(s - p_k^*)} \qquad (4)$$

where,

$m =$ Number of modes of vibration

$[r_k] =$ (**n by n**) residue matrix for the k^{th} mode

$p_k = -\sigma_k + j\omega_k =$ Pole location for the k^{th} mode

$\sigma_k =$ Damping decay of the k^{th} mode

$\omega_k =$ Natural frequency of the k^{th} mode

$\{u_k\} =$ Mode shape for the k^{th} mode (n-vector)

$A_k =$ Scaling constant for the k^{th} mode

Figure 1 shows a transfer function for a single mode of vibration, plotted over ***half of the s-plane***.

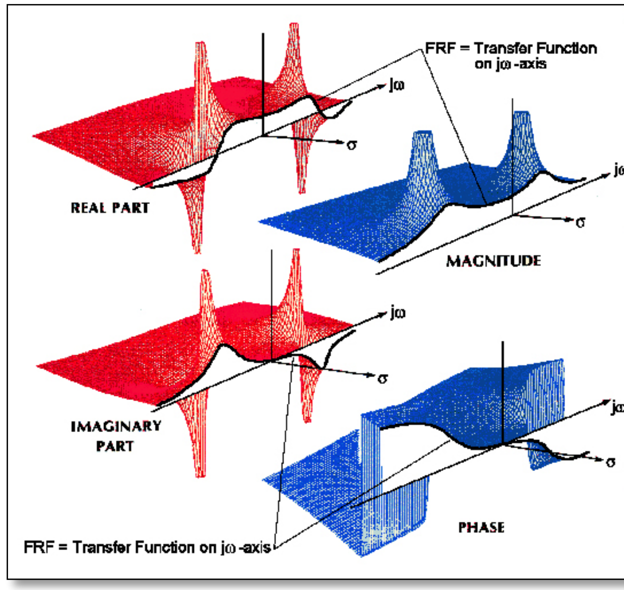

Figure 1. Transfer Function & FRF for a Single Mode in the s-plane

Experimental FRFs

An FRF is defined as the values of a transfer function along the **$j\omega$-axis** in the s-plane, as shown in Figure 1. FRFs can only be calculated from experimental data when ***all of the excitation forces and responses*** caused by them are ***simultaneously acquired***. Equation (3) is the analytical form of an FRF that is used for ***FRF-based curve fitting*** of experimental data. The outcome is an EMA pole & mode shape for each term that is used in the summation during curve fitting.

Mode Shape Curve Fitting

Equation (4) also tells us that each FRF can be represented as a ***summation of resonance curves***, hence a set of FRFs can be ***decomposed*** into their resonance curves by ***curve fitting them one frequency at a time*** with a set of mode shapes.

Output-Only Frequency Spectra

In cases where the excitation forces are not measured and therefore FRFs cannot be calculated, three other types of ***output-only*** frequency spectra can be calculated from acquired response time waveforms; Fourier spectra, Cross spectra, and ODSFRFs.

A **Fourier spectrum** is simply the Digital Fourier Spectrum (DFT) of a digital response time waveform. It is calculated with the FFT algorithm. A **Cross spectrum** is calculated between ***two simultaneously acquired responses***. It is the DFT on one signal multiplied by the complex conjugate of the other. An **ODSFRF** is the Auto spectrum of a ***roving*** response combined with the phase of the Cross spectrum between the ***roving*** response and a ***(fixed) reference*** response.

When it can be assumed that these output-only measurements can also be represented as a ***summation of resonance curves***, they too can be ***decomposed*** into a set resonance curves by ***curve fitting them one frequency at a time*** with a set of mode shapes.

Operating Deflection Shape (ODS)

When a vibration response is analytically modeled or experimentally measured at ***two or more points & directions*** on a machine or structure, this data is called an **Operating Deflection Shape** (or **ODS)** [4]. Three different types of ODS's are possible; ***time-based ODS's***, ***frequency-based ODS's***, and ***order-based ODS's***.

Time-Based ODS

A time-based ODS is simply the values of two or more simultaneously acquired or calculated time waveforms for the same time value. A time-based ODS is the ***true overall response*** of the structure at any moment of time.

Frequency-Based ODS

A frequency-based ODS is the values of two or more frequency domain functions (FRFs or spectra) at the same frequency. A frequency-based ODS is the ***true overall response*** of the structure for any frequency for which the measurement functions were calculated

All frequency domain functions, (except an Auto spectrum) are complex valued (with magnitude & phase), so all frequency-based ODS's are also complex valued.

Order-Based ODS

In a rotating machine, the dominant forces are applied at multiples of the machine running speed, called orders. An ***order-based ODS*** is assembled from the ***peak values*** at one of the order frequencies in a set of output-only frequency spectra. These spectra are calculated from response data that is acquired while the machine is running. When displayed in animation on a 3D model of the machine, an order-based ODS is a convenient way of visualizing distributed vibration levels. These distributed levels can also be used for monitoring the health of the machine.

ODS Expansion

In a previous paper [1], it was shown how ***modes participate*** in an order-based ODS of a rotating machine, and how they participate differently at different operating speeds. It another previous paper [2], it was also shown how the ***modal participation*** can be used to ***expand*** an order-based ODS so that it is a valid representation of the ODS for all DOFs of the machine, both ***measured*** & ***un-measured***.

Measurement Expansion

In this paper the same curve fitting and expansion equations that were used for ODS expansion will be used to decompose and expand a set of FRFs, and sets of output-only Cross spectra and ODSFRFs.

EXPANDING EMA MODES USING FEA MODES

The aluminum plate shown in Figure 2 was tested using a roving impact hammer. During the test, a 5 by 6 grid of points was impacted in the vertical direction. A tri-axil accelerometer located near one corner of the plate was used to measure the response due to the impacts.

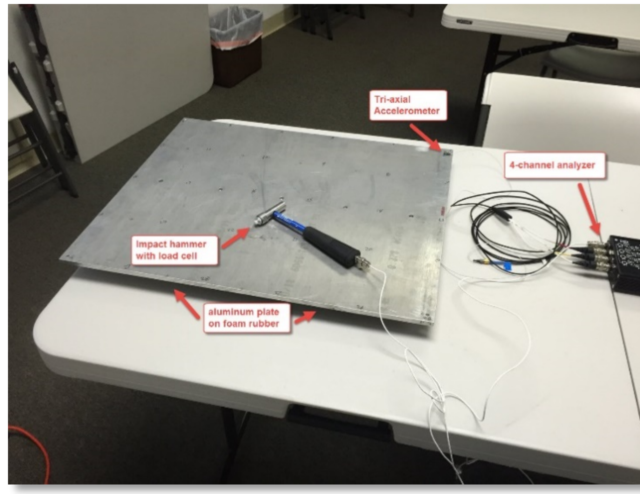

Figure 2. Impact Test of Aluminum Plate

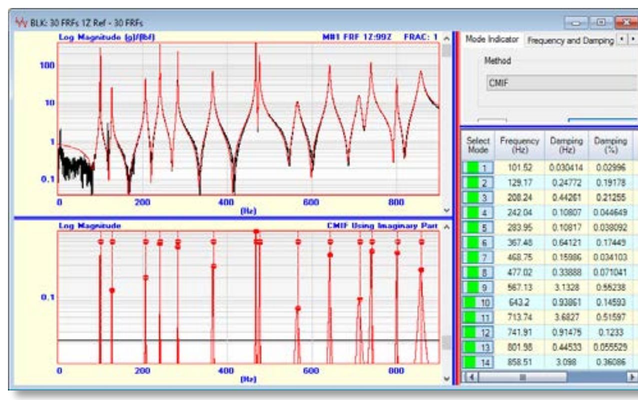

Figure 3. FRF-based Curve Fit of a Measurement

FRFs were calculated from simultaneously acquired force & response data, and the modal parameters of 14 EMA modes were extracted by curve fitting the FRFs using FRF-based curve fitting. A typical curve fit is shown in Figure 3.

Plate FEA Model

Also, an FEA model of the plate was created by adding 180 FEA brick elements to the same model used to display the EMA mode shapes. The first flexible body FEA mode is shown in Figure 4. The 30 impact points are also labeled.

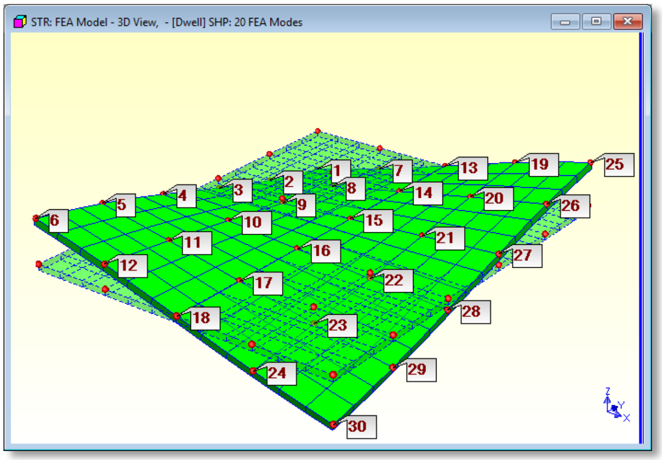

Figure 4. FEA Mode Shape in Animation

Modal Participation

The bar chart of the participations of the first 14 flexible body FEA modes in the 14 EMA modes is shown in Figure 5. The participation values reflect the different scaling of the FEA mode shapes versus the EMA modes. The FEA mode shapes were scaled to ***Unit Modal Masses***, while the EMA mode shapes contained the ***residues (numerators)*** obtained from using equation (3) to curve fit the FRFs.

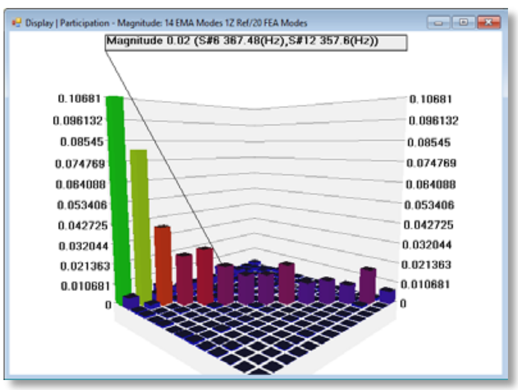

Figure 5. Participation of FEA modes in EMA modes

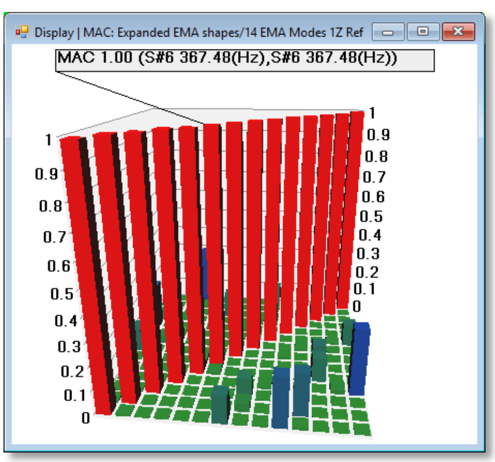

Figure 6. MAC Values between Expanded & Original EMA modes

Shape Expansion

The participation factors in Figure 5 were used to expand the EMA modes from 30 DOFs to 1248 DOFs, the number of FEA mode shape DOFs. The MAC values between the expanded & original EMA mode shapes are shown in Figure 6. The SDI values between the expanded EMA and the original EMA mode shapes are shown in Figure 7.

Both Figures 6 & 7 indicate that the FEA mode shapes were accurately curve fit to the EMA shapes. MAC indicates the ***co-linearity*** of each shape pair. SDI more strongly indicates that each shape pair has ***nearly the same values*** for the 30 DOFs that are common among the original and expanded shapes.

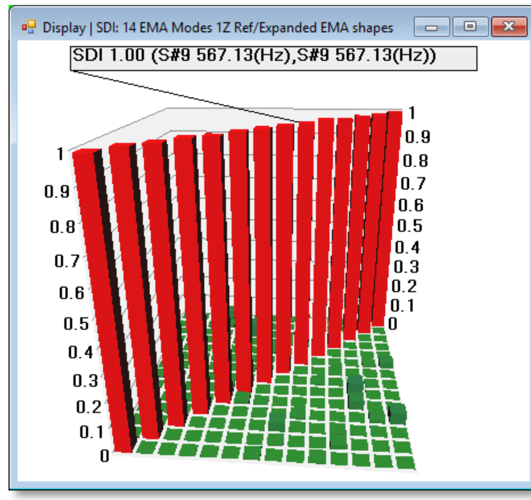

Figure 7. SDI Values between Expanded & Original EMA modes

FRF DECOMPOSITION

The 14 FEA mode shapes were then used to decompose the 30 FRF measurements for the aluminum plate at each frequency. The values of the 30 FRFs at each frequency are a ***frequency-based ODS***. The FEA mode shapes participate differently in the ODS at each frequency, and when the participations are plotted for all frequencies, they create a separate resonance curve for each FEA mode. Figure 8 is a plot of the 14 resonance curves obtained from decomposing the 30 FRFs using the FEA modes.

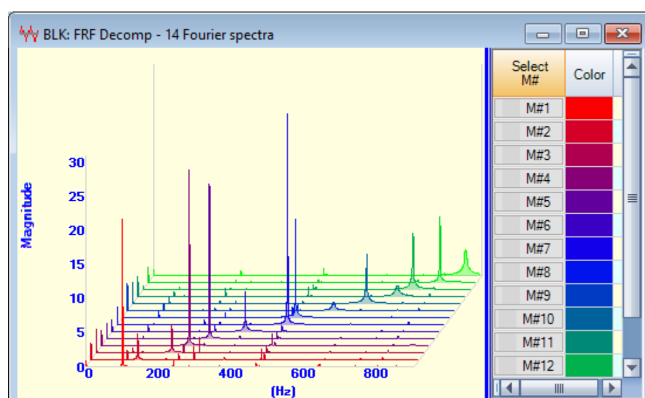

Figure 8. FRF Decomposition using 14 FEA modes

Notice that each resonance curve in Figure 8 has a ***single dominate peak*** in it at the natural frequency of one of the 14 resonances. These resonance curves clearly illustrate the superposition property of modes.

FRF RECONSTRUCTION

A set of ***reconstructed FRFs*** can then be calculated by multiplying the resonance curves in Figure 8 by the FEA mode shapes. Figure 9 shows a reconstructed FRF overlaid on its corresponding original experimental FRF.

FRF EXPANSION

Not only are the original 30 FRFs reconstructed, but an ***expanded set of 1248 FRFs*** can be calculated using all the DOFs of the FEA mode shapes. This expanded set of FRFs can then be curve fit using FRF-based curve fitting to obtain a set of EMA modes with ***frequency, damping***, and ***mode shapes with 1248 DOFs*** in them.

NOTE: ***Only FEA mode shapes*** are required to perform decomposition, reconstruction, and expansion of experimental vibration data.

Decomposition, reconstruction and expansion can be applied to either time or frequency domain vibration data.

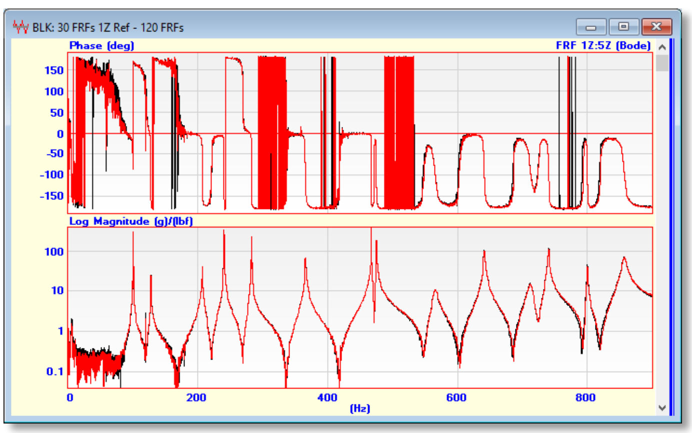

Figure 9. Reconstructed & Experimental FRFS Overlaid

CROSS SPECTRUM DECOMPOSITION

The 14 FEA mode shapes were also used to decompose 30 Cross spectrum measurements for the aluminum plate at each frequency. Figure 10 is a plot of the participations of the 14 FEA modes in the Cross spectra measurements.

CROSS SPECTRUM EXPANSION

Figure 11 shows a reconstructed Cross spectrum overlaid on an original experimental Cross spectrum. The reconstructed Cross spectra were calculated by multiplying the resonance curves in Figure 10 by the FEA mode shapes. Again, not only can the original 30 Cross spectra be reconstructed, but an ***expanded set of 1248 Cross spectra*** can be calculated using the DOFs of the FEA mode shapes.

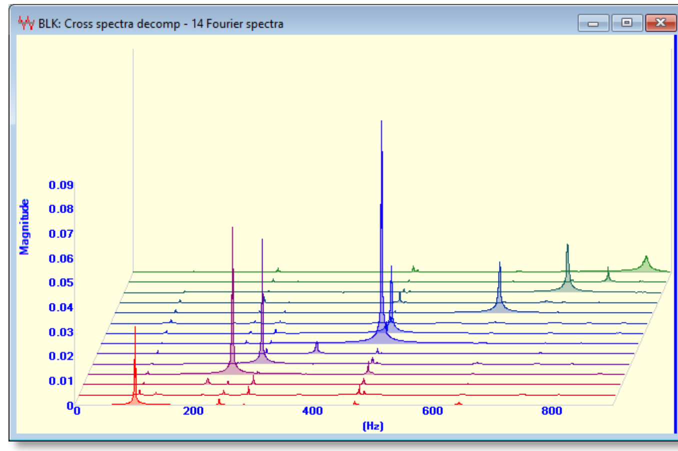

Figure 10. Cross spectra Decomposition using 14 FEA modes

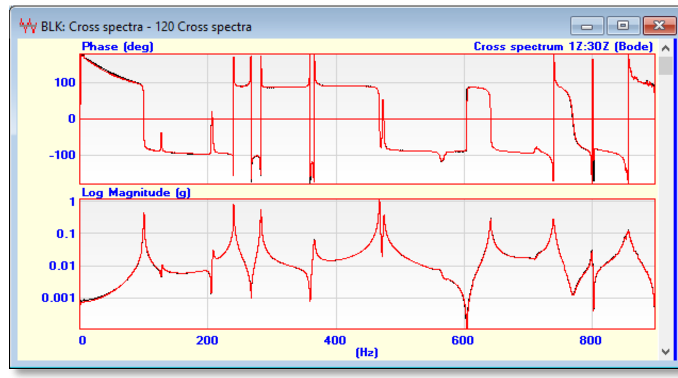

Figure 11. Reconstructed & Experimental Cross spectra Overlaid

ODSFRF DECOMPOSITION

The 14 FEA mode shapes were also used to decompose 30 ODS-FRF measurements for the aluminum plate at each frequency. Figure 12 is a plot of the participations (or resonance curves) of the 14 FEA modes in the ODSFRF measurements.

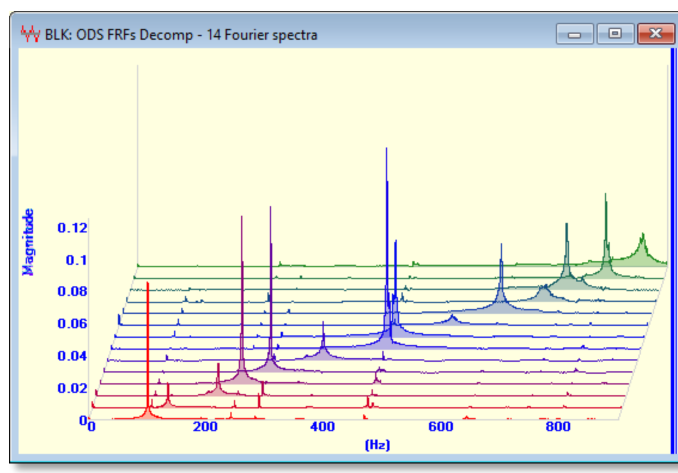

Figure 12. ODSFRF Decomposition using 14 FEA modes

ODSFRF EXPANSION

Figure 12 shows a reconstructed ODSFRF overlaid on an original experimental ODSFRF. The reconstructed ODFRFs were calculated by multiplying the resonance curves in Figure 12 by the FEA mode shapes. Again, not only can the original ODSFRFs be reconstructed, but an ***expanded set of 1248 ODSFRFs*** can be calculated using all DOFs of the FEA mode shapes.

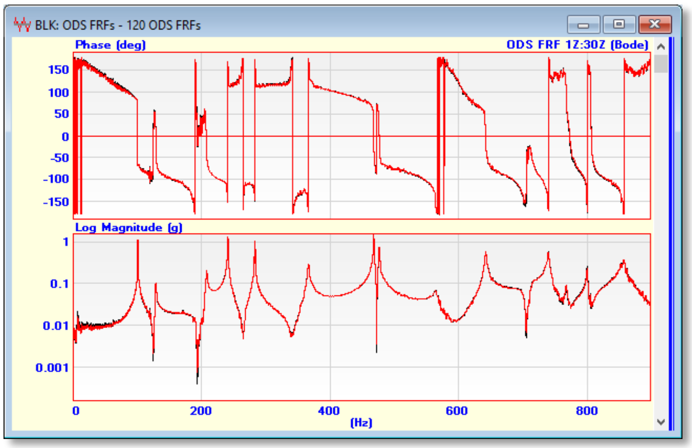

Figure 13. Reconstructed & Experimental ODS FRFs Overlaid

MEASUREMENT ERRORS

One of the most useful applications of measurement curve fitting and expansion using mode shapes is that experimental errors are quickly spotted. Bad measurements are found by overlaying each reconstructed measurement on its corresponding experimental data. One pair of each of the measurement types is overlaid in Figures 8, 11, & 13.

Measurement Comparison Using MAC & SDI

A numerical method for comparing reconstructed & experimental data is to use MAC & SDI values. Both MAC & SDI have values between 0 & 1. When MAC is applied to pairs of FRFs, it is also called the ***Frequency Response Assurance Criterion*** (or ***FRAC***).

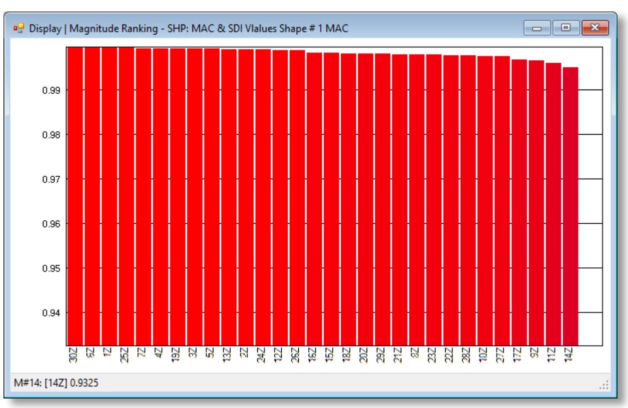

Figure 14. Ordered MAC for Reconstructed & Experimental FRF Pairs

Figure 14 is a magnitude plot of the MAC values between the 30 reconstructed & experimental FRFs, ordered from the highest to lowest MAC value. Figure 15 is an ordered magnitude plot of SDI values.

Both MAC & SDI have their lowest values for the reconstructed & experimental FRF pair at DOF 14Z. These low values indicate a mismatch between the reconstructed & experimental FRFs. The overlaid reconstructed & experimental FRF pair is shown in Figure 16.

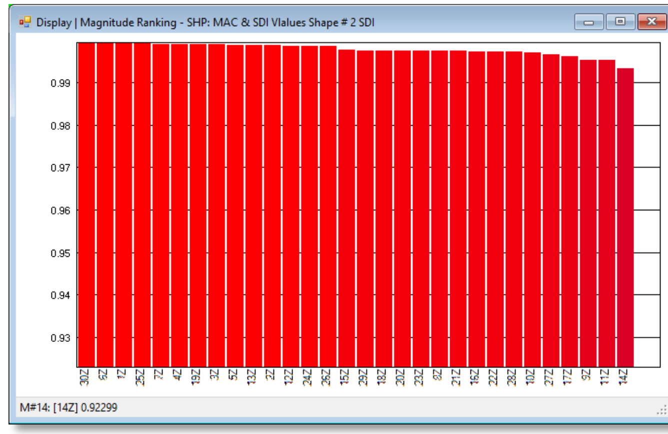

Figure 15. Ordered SDI for Reconstructed & Experimental FRF Pairs

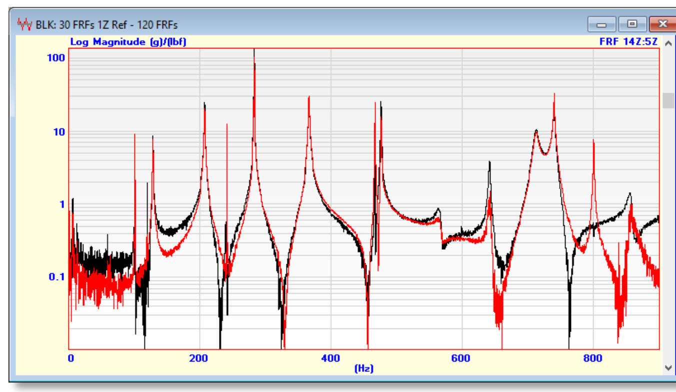

Figure 16. Reconstructed & Experimental 14Z:5Z FRFs Overlaid

Since all of the other FRF pairs have high MAC & SDI values (close to 1), this is strong evidence that curve fitting the FEA mode shapes to the experimental data provided accurate reconstructing FRFs, except in a few cases like the pair shown in Figure 16. Since the reconstructed FRFs were the result of a ***least-squared-error curve fit*** to all 30 experimental FRFs, the strongest conclusion from the mismatch in Figure 16 is that the ***experimental FRF data is in error.***

Since the aluminum plate was tested using as roving impact test, it can be concluded that Point 14 on the plate was ***impacted at a different point*** than Point 14 on the FEA model.

CONCLUSIONS

It was shown how the FEA mode shapes of a structure can be used to decompose, reconstruct, and expand a set of FRFs, Cross spectra, and ODSFRFs. FRFs are normally calculated when all of the excitation forces causing a structure to vibrate are measured. Cross spectra & ODSFRFs are calculated from ***output-only data*** when the excitation forces are not measured.

It was demonstrated in each case that these three types of experimental data can be decomposed into a ***summation of resonance***

curves by curve fitting a set of mode shapes to them, one frequency at a time. Then it was shown that the participations of the modes in the experimental data can be used to construct an ***expanded set of measurements*** using the DOFs of the mode shapes, including the DOFs that were common with the experimental data.

A key advantage of this approach is that ***only the mode shapes*** are required in these calculations. Accurate mode shapes can be easily obtained from a simple FEA model. Accurate FEA frequencies that match the EMA frequencies usually require a more accurate FEA model, but frequencies are not required for this calculation.

Another advantage of this approach is that complex data can be decomposed using ***normal modes***. Modes are called ***normal*** when the FEA model they are derived from has ***no damping terms***, and hence the mode shapes are real valued or normal. Normal modes can be used to expand ***complex mode shape or ODS data*** because the ***modal participation factors are complex*** valued.

This decomposition and expansion capability is useful not only for creating measurements for all of the ***un-measured DOFs*** on a structure, but also for identifying bad measurements. The expanded set of measurements can also be curve fit using FRF-based curve fitting to obtain EMA modes with ***frequency, damping***, and ***expanded mode shapes*** in them. This EMA modal model can then be used for SDM and MIMO Modeling & Simulation studies involving ***un-measured DOFs*** of the structure.

This combined use of an analytical model with experimental data provides a more complete characterization of the dynamic behavior of a structure from a ***relatively small number of measurements***. This means that less time & expense are required to obtain meaningful data for use in machinery & structural health monitoring, or for troubleshooting noise & vibration problems.

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